Exercise 1

Let $A$ and $B$ be two $n \times n$ matrices over an arbitrary field.

1. Prove that
   \[ \text{rk}(A + B) \leq \text{rk}(A) + \text{rk}(B). \]  

2. Show an example in which inequality (1) is strict.

Exercise 2

Let $A$ be an $n \times n$ matrix with entries in $\mathbb{R}$ and let $x \in \mathbb{R}^n$ be a column vector. Prove that $x^T A^T x = x^T A x$.

Exercise 3

Let $G$ be a graph with vertex set $V = \{1, 2, \ldots, n\}$. Let $A$ be the adjacency matrix of $G$. Prove that, for every $1 \leq i, j \leq n$ and for every positive integer $k$, the $(i, j)$th entry of $A^k$ is the number of walks (an edge can be traversed more than once) of length $k$ between $i$ and $j$ in $G$.

Exercise 4

Let $B_1, B_2, \ldots, B_m \subseteq K_n$ be complete bipartite graphs that cover every edge of $K_n$ an odd number of times. Prove that $m \geq (n - 1)/2$. 