Exercise 1

Fix a field $\mathbb{F}$. All vector spaces and (multi)linear maps will be over $\mathbb{F}$. Let $V$ be an $n$-dimensional vector space and let $0 \leq k \leq n$.

1. If $f : V^k \to W$ is an arbitrary alternating $k$-multilinear map into a vector space $W$, then there exists a unique linear map
   $$\tilde{f} : \bigwedge^k V \to W$$
   such that for all $v_1, \ldots, v_k \in V$,
   $$f(v_1, \ldots, v_k) = \tilde{f}(v_1 \wedge \ldots \wedge v_k).$$
   This is called the universal property of the exterior product.

2. Show that the universal property uniquely determines $\bigwedge^k V$. That is, suppose $U$ is a vector space and $\varphi : V^k \to U$ is an alternating $k$-linear map such that for every $f$ as above, there is a unique linear $f : U \to W$ such that $f(v_1, \ldots, v_k) = f(\varphi(v_1, \ldots, v_k))$. Show that there is an isomorphism $h : \bigwedge^k V \to U$ such that $h(v_1 \wedge \ldots \wedge v_k) = \varphi(v_1, \ldots, v_k)$ for all $v_1, \ldots, v_k \in V$.

Exercise 2

Let $\mathbb{F}$ be a field. For $x \in \mathbb{F}$ and $d \geq 1$, let $\gamma(x) = (1, x, \ldots, x^{d-1})^T \in \mathbb{F}^d$. Show that for all $x_1, \ldots, x_d \in \mathbb{F}$,
   $$\det(\gamma(x_1) | \ldots | \gamma(x_d)) = \prod_{1 \leq i < j \leq d} (x_i - x_j).$$

Exercise 3

A hypergraph $H = (V, \mathcal{F})$ is called a star if for every hyperedge $F \in \mathcal{F}$ there exists a vertex $v \in V$ that belongs only to $F$, that is, $v \in F$ and $v \notin G$ for any $G \in \mathcal{F}$. Let $r, t$ be positive integers.

1. Let $H = (V, \mathcal{F})$ be a finite hypergraph such that $H$ does not contain a star with $t + 1$ hyperedges. Show that $H$ has a transversal $T \subseteq V$ of size $t$. (A set $T$ of vertices is a transversal if $T \cap F \neq \emptyset$ for every nonempty $F \in \mathcal{F}$.) Show that the same holds for all hypergraphs $H \setminus Y = (V \setminus Y, \mathcal{F}|_{V \setminus Y})$, $\mathcal{F}|_{V \setminus Y} = \{F \setminus Y : F \in \mathcal{F}\}$, where $Y \subseteq V$ is arbitrary.

2. Let $H$ be as above and of rank $r$. That is, $|F| \leq r$ for all $F \in \mathcal{F}$. Show that $|\mathcal{F}| \leq \binom{r+t}{r}$.

3. Give an example of a hypergraph as above with $|\mathcal{F}| = \binom{r+t}{r}$. 