Algebraic Methods in Combinatorics

Exercise 1

Let $A$ be an $n \times n$ matrix with entries in some field $\mathbb{F}$. Prove that if $\text{Per}(A) \neq 0$ (in $\mathbb{F}$), then, for any subsets $S_i \subseteq \mathbb{F}$, $1 \leq i \leq n$, each of size 2, there exists a vector $x \in S_1 \times \ldots \times S_n$ such that no entry of $Ax$ is zero (that is, $Ax \in (\mathbb{F} \setminus \{0\})^n$).

Exercise 2

The bandwidth of a graph $G = (V, E)$ on $n$ vertices is the smallest integer $t$ for which there exists a bijection $f : V \to \{1, 2, \ldots, n\}$ such that $|f(u) - f(v)| \leq t$ for every edge $(u, v) \in E$. Prove that the bandwidth of $G$ is at least $k + 1$ if and only if the polynomial

$$Q^k_G(x_1, \ldots, x_n) := \prod_{1 \leq i < j \leq n} (x_i - x_j) \prod_{(v_i, v_j) \in E} \prod_{k < t \leq n} (x_i - x_j - t)(x_i - x_j + t)$$

lies in the ideal which is generated by the polynomials $\prod_{i=1}^{n} (x_i - j) : 1 \leq i \leq n$.

Remark: Consider the polynomials to be elements of the ring $\mathbb{R}[x_1, \ldots, x_n]$ (this choice is quite arbitrary).

Exercise 3

Let $H = (V, E)$ be a 4-uniform hypergraph. Define a polynomial $P$ and a set of polynomials $\{g_i : i \in I\}$ such that the following claim will hold (prove that it does): $H$ is not 2-colorable if and only if $P$ lies in the ideal which is generated by the polynomials $\{g_i : i \in I\}$.

Remark: There are many possible choices for $P$ and the $g_i$'s.