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Algebraic Methods in Combinatorics — *HS 08*

Exercise Set 9

Exercise 1

Let A be an $n \times n$ matrix with entries in some field $\mathbb{F}$. Prove that if $\mathrm{Per}(A) \neq 0$ (in $\mathbb{F}$), then, for any subsets $S_i \subseteq \mathbb{F}$, $1 \leq i \leq n$, each of size 2, there exists a vector $x \in S_1 \times \ldots \times S_n$ such that no entry of Ax is zero (that is, $Ax \in (\mathbb{F} \setminus \{0\})^n$).

Exercise 2

The *bandwidth* of a graph $G = (V, E)$ on n vertices is the smallest integer t for which there exists a bijection $f : V \to \{1, 2, \ldots, n\}$ such that $|f(u) - f(v)| \leq t$ for every edge $(u, v) \in E$. Prove that the bandwidth of G is at least $k + 1$ if and only if the polynomial

$$Q_G^k(x_1, \ldots, x_n) := \prod_{1 \leq i < j \leq n} (x_i - x_j) \prod_{\substack{(v_i, v_j) \in E \\ i < j}} \prod_{k < t < n} (x_i - x_j - t)(x_i - x_j + t)$$

lies in the ideal which is generated by the polynomials $\{\prod_{j=1}^n (x_i - j) : 1 \leq i \leq n\}$.

Remark: Consider the polynomials to be elements of the ring $\mathbb{R}[x_1, \ldots, x_n]$ (this choice is quite arbitrary).

Exercise 3

Let $H = (V, E)$ be a 4-uniform hypergraph. Define a polynomial P and a set of polynomials $\{g_i : i \in I\}$ such that the following claim will hold (prove that it does): H is not 2-colorable if and only if P lies in the ideal which is generated by the polynomials $\{g_i : i \in I\}$.

Remark: There are many possible choices for P and the g_i's.