Institute of Theoretical Computer Science
Emo Welzl, Thomas Holenstein

September 16, 2009

Algorithms, Probability, and Computing *Fall 09*
Preparatory Exercise

The following exercises will be solved in the first exercise class on September 16, 2009.

Exercise 1

Let D be a biased die and let X be the random variable for a roll of D. The following table gives the probability distribution for X

X	1	2	3	4	5	6
probability	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{3}{8}$

(a) Compute $E[X]$

(b) Suppose that we throw two dice D_1, D_2 (both having the probability distribution given by the above table).

(i) What is the expectation value of their sum?

(ii) What is the expectation value of their product?

(iii) Compute the probability that the sum of D_1 and D_2 is at most 4

Exercise 2

We now consider two fair dice (i.e. every number appears with probability $\frac{1}{6}$). Let X_1 and X_2 denote the random variable for the outcome of the first die and the second die, respectively.

(a) Compute $\Pr[X_1 + X_2 = 8]$

(b) Compute $\Pr[X_1 + X_2 \geq 6 | X_1 \leq 2]$

(c) Determine for each of the following pairs $\mathcal{E}_1, \mathcal{E}_2$ of events whether they are dependent

(i)
- $\mathcal{E}_1$: X_1 is even.
- $\mathcal{E}_2$: $X_1 + X_2$ is odd.

(ii)
- $\mathcal{E}_1$: X_1 is even.
- $\mathcal{E}_2$: $X_1 + X_2 \geq 8$.

(iii)
- $\mathcal{E}_1$: $X_1 = X_2$.
- $\mathcal{E}_2$: $X_1 + X_2 \geq 10$.

(iv)
- $\mathcal{E}_1$: $X_1 \geq X_2$.
- $\mathcal{E}_2$: $X_1 + X_2 \leq 3$.

Exercise 3

(a) In the famous *Monty Hall Problem*, you are on a TV show, facing three closed doors on the stage. You know that behind two doors there is a goat, and behind one there is a beautiful car. The rules are as follows: In step 1, you point at some door. In step 2, the showmaster opens one of the remaining doors, but he is not allowed to open the door with the car behind it. In step 3 you can point to one of the closed doors, and get as a prize whatever is behind it.

To maximize your chances of getting the car, should you in step 3 stay at your door or switch to the other closed door?

(b) Harry and Hermione are in the laboratory of an evil wizard. On the desk, there are three cups with a potion in them. Harry and Hermione know that two of them are poisonous, but one gives unlimited power. They also know that the evil wizard will return soon and kill them if they do not drink the "good" potion. Harry randomly chooses a cup and lifts it to his lips (not yet drinking). Hermione chooses randomly one of the other two, drinks it—and dies. What should Harry do? Should he drink from the cup he is holding, or should he switch to the third cup, or does it not matter? Note that running away is not an option. Besides being futile, it would be unfair to Hermione.

Exercise 4

(a) Given a fair coin, what is the probability that we have to throw it an even number of times until head appears for the first time (i.e. what is the probability that we obtain a sequence of the form $TH, TTTH, TTTTTH, \ldots$ with T denoting tail and H denoting head) ?

(b) Suppose we are given a biased coin which lands head with probability $\frac{1}{4}$. What is now the probability that we have to throw it an even number of times until head appears for the first time?

Exercise 5

Consider a gambling game in which you toss a fair coin C. If C lands head, you multiply your stake by 2 otherwise you multiply your stake by $\frac{1}{4}$. Suppose that you start with 10 Fr.

(a) What is the expected amount after n rounds?

(b) What is the probability that after 3 rounds you have more than 20 Fr.?

(c) Let p denote the probability that after 20 rounds you have at least 10 Fr. Is p larger or smaller than 0.5?

Exercise 6

An apple is located at vertex A of pentagon $ABCDE$, and a worm is located two vertices away, at C. Every day the worm crawls with equal probability to one of the two adjacent vertices. Thus after one day the worm is at vertex B or D, each with probability $\frac{1}{2}$. After two days, the worm might be back at C again, because it has no memory of previous positions. When it reaches vertex A, it stops to dine.

(a) What is the mean of the number of days until dinner?

(b) Let p be the probability that the number of days is 100 or more. What does Markov's Inequality say about p?

Exercise 7

Give an example of a probability space with three events A, B, C such that the following holds:

(a) The events are pairwise independent, i.e.

$$\begin{aligned} \Pr[A \cap B] &= \Pr[A] \cdot \Pr[B] \\ \Pr[B \cap C] &= \Pr[B] \cdot \Pr[C] \\ \Pr[A \cap C] &= \Pr[A] \cdot \Pr[C] \end{aligned}$$

(b) The events are *not* independent, i.e.

$$\Pr[A \cap B \cap C] \neq \Pr[A] \cdot \Pr[B] \cdot \Pr[C]$$