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December 4, 2009

Algorithms, Probability, and Computing *Fall 09*
Special Problem Set 3

Read the paper "A Simple Min-Cut Algorithm" by Stoer and Wagner of which we posted a link on the webpage.

Cuts and Flows in Undirected Graphs. In the lecture, we have defined s–t-flows and s–t-cuts for directed graphs with edge capacities. Here, we need to adapt these concepts to undirected graphs. An undirected network is a pair (G, c), with $G = (V, E)$ being an undirected graph and $c : E \to \mathbb{R}_0^+$. For two disjoint sets $A, B \subseteq E$ we write

$$c(A, B) := \sum_{a \in A, b \in B: \{a,b\} \in E} c(\{a, b\}) \ .$$

An s–t-flow in this network is a function $f : V \times V \to \mathbb{R}_0^+$ satisfying

- $f(u, u) = 0$ for all $u \in V$,
- $f(u, v) \leq c(\{u, v\}) \quad \forall (u, v) \in V \times V$,
- $NetOut_f(u) = 0$ for every vertex $u \in V \setminus \{s, t\}$,

where $NetOut_f(u) := \sum_{v:\{u,v\} \in E} f(u, v) - f(v, u)$. The *value* of f is defined to be $NetOut_f(s)$.

Exercise 1

Using these notions, state and prove a maxflow-mincut theorem for undirected graphs with edge capacities. Remark: No formal proof is required. Just show how to use the directed version of the maxflow-mincut theorem! Your solution should not exceed half a page.

Exercise 2

Prove the following lemma.

Lemma 1. *Let G be an undirected graph with edge capacities, and a, b, c be vertices in G. Suppose there exist an a–b-flow and a b–c-flow, both of value x. Then there exists an a–c-flow of value x as well.*

Remark: don't try to do anything complicated. There is a three-line solution. Your solution must not be longer than half a page.

Exercise 3

We want to give an alternative proof of Lemma 3.1. Let $v_1, v_2, \ldots, v_n$ be the vertices of G in the order in which they have been added to A in the procedure MINIMUMCUTPHASE. For disjoint sets $A, B \subseteq V(G)$, let $c(A, B)$ denote the sum of the capacities of the edges between A and B.

Lemma 2. *For all $1 \leq i < j \leq n$, there exists a v_i–v_j-flow in the induced subgraph $G[\{v_1, \ldots, v_i, v_j\}]$ of value at least $c(\{v_1, \ldots, v_i\}, \{v_j\})$.*

Prove the lemma. Hint: Use induction on i, and at some point use Lemma 1. Show that it implies Lemma 3.1 from the paper. If you don't know what an induced subgraph above is: for a graph $G = (V, E)$ and a set $U \subseteq V$ of vertices, we define the subgraph of G induced by U by $G[U] := \left(U, E \cap \binom{U}{2}\right)$, i.e. the graph consisting of U and the edges of E connecting vertices in U.

Exercise 4

Suppose you run the algorithm MinimumCutPhase on a directed graph. Explain how you change the notion of "most tightly connected" (below the description of the algorithm, page 587). Show that the algorithm does not work on directed graphs: Give an example of a directed graphs (with edge capacities) on which $(\{v_1, \ldots, v_{n-1}\}, \{v_n\})$ is not a minimum v_{n-1}–v_n-cut.

Exercise (Challenge!)

Lemma 3.1 immediately implies the following surprising statement: Every undirected graph with edge capacities contains two vertices s and t such that the cut $(V \setminus \{t\}, \{t\})$ is a minimum s–t-cut (this is clear: simply consider v_{n-1} and v_n, the two last added vertices). Is the statement true for *directed* graphs as well? If not, a counter-example would be a directed graph $G = (V, E)$ such that for all pairs $(s, t) \in V \times V$, the cut $(V \setminus \{t\}, \{t\})$ is *not* a minimum s–t-cut.

Remark: So far, I don't have a solution for this problem. I personally guess the answer is *no*, but I did not find a counter-example yet and did not search existing literature.