General rules for solving exercises

- This is a theory course, which means: if an exercise does not explicitly say “you do not need to prove your answer” or “justify intuitively”, then a formal proof is always required.

- All exercises and their solutions, no matter whether they are graded or regular/optional ones, are part of the material relevant for the two exams.

- Some of the exercises are marked as "in-class", which means that we do not expect you to solve them before the exercise session. Instead, your teaching assistant will solve them with you in class.

- You are highly encouraged to solve all other exercises (those not marked as "in-class") on your own and to hand in a writeup of your solutions no later than the due date. If you choose to do so, please write the name of your teaching assistant on the front sheet.

The following exercises will be solved in the first exercise class on September 17, 2014. Since all exercises are in-class, there is no due date.

Exercise 1: Expectation (in-class)

Let $D$ be a biased die and let $X$ be the random variable for a roll of $D$. The following table gives the probability distribution for $X$.

<table>
<thead>
<tr>
<th>$X$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>probability</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{3}{8}$</td>
</tr>
</tbody>
</table>

(a) Compute $E[X]$ and $E[X^2]$

(b) Suppose that we throw two dice $D_1$, $D_2$ (both having the probability distribution given by the above table).

(i) What is the expectation value of their sum?

(ii) What is the expectation value of their product?

(iii) Compute the probability that the sum of $D_1$ and $D_2$ is at most 4.

Exercise 2: Dependency (in-class)

We now consider two fair dice (i.e. every number appears with probability $\frac{1}{6}$). Let $X_1$ and $X_2$ denote the random variable for the outcome of the first die and the second die, respectively.

(a) Compute $Pr[X_1 + X_2 = 8]$
(b) Compute $\Pr [X_1 + X_2 \geq 6 \mid X_1 \leq 2]$

(c) Determine for each of the following pairs $\mathcal{E}_1, \mathcal{E}_2$ of events whether they are dependent:

(i)  
- $\mathcal{E}_1$: $X_1$ is even.
- $\mathcal{E}_2$: $X_1 + X_2$ is odd.

(ii)  
- $\mathcal{E}_1$: $X_1$ is even.
- $\mathcal{E}_2$: $X_1 + X_2 \geq 8$.

(iii)  
- $\mathcal{E}_1$: $X_1 = X_2$.
- $\mathcal{E}_2$: $X_1 + X_2 \geq 10$.

(iv)  
- $\mathcal{E}_1$: $X_1 \geq X_2$.
- $\mathcal{E}_2$: $X_1 + X_2 \leq 3$.

Exercise 3: Conditional Probability (in-class)

(a) Suppose you have an egg and a bottle of milk in your fridge. Each of them is spoiled with probability exactly 0.5, independently of the other. Assume that you inspect the milk and find out that it is spoiled. What is now the probability that the egg is spoiled as well?

(b) Suppose a new family moves into your neighborhood. All you know about them is that they have two children. At some point you get the information that one of the two children is called Markus. What is the probability that the other child is a boy as well?

For simplicity, we assume that each child is a boy with probability exactly 0.5, independently of its siblings.

Exercise 4: Paradoxes (in-class)

(a) In the famous Monty Hall Problem, you are on a TV show, facing three closed doors on the stage. You know that behind two doors there is a goat, and behind one there is a beautiful car. The rules are as follows: In step 1, you point at some door. In step 2, the show master opens one of the remaining doors, but he is not allowed to open the door with the car behind it. In step 3 you can point to one of the closed doors, and get as a prize whatever is behind it.

Now assume that you have picked door 1 in step 1, the car is behind a door uniformly at random, and the show master picks uniformly at random if he has the choice. You see the show master has opened door 2. Should you now pick door 1 or 3?

(b) Now assume the show master does not pick uniformly at random. If the car is behind door 1, he picks door 2 with probability $p$ and door 3 with probability $1 - p$. What is your optimal strategy now if the show master opens door 2? What if he opens door 3? What is the overall probability to get the car?

(c) Harry and Hermione are in the laboratory of an evil wizard. On the desk, there are three cups with a potion in them. Harry and Hermione know that two of them are poisonous, but one gives unlimited power. They also know that the evil wizard will return soon and kill them if neither of them drinks the "good" potion. Harry randomly chooses a cup and lifts it to his lips (not yet drinking). Hermione chooses randomly one of the other
two, drinks it — and dies. What should Harry do? Should he drink from the cup he is holding, or should he switch to the third cup, or does it not matter? Note that running away is not an option.

Exercise 5: Geometric Distributions (in-class)

(a) Given a fair coin, what is the probability that we have to throw it an even number of times until head appears for the first time (i.e. what is the probability that we obtain a sequence of the form TH, TTH, TTTTH, . . . with T denoting tail and H denoting head)?

(b) Suppose we are given a biased coin which lands head with probability \( \frac{1}{4} \). What is now the probability that we have to throw it an even number of times until head appears for the first time?

Exercise 6: Gambler’s Ruin (in-class)

Consider a gambling game in which you toss a fair coin \( C \). If \( C \) lands head, you multiply your stake by 2 and otherwise you multiply your stake by \( \frac{1}{4} \). Suppose that you start with CHF10.

(a) What is the expected amount after \( n \) rounds?

So far, does it look like this is a game you should agree to play?

(b) What is the probability that after 3 rounds you have more than CHF20?

(c) Let \( p_n \) denote the probability that after \( n \) rounds you have at least CHF10. Give a formula for \( p_n \).

(d) What happens to \( p_n \) if \( n \to \infty \)? HINT: Use the following approximation formula:

\[
\sum_{k=0}^{\lfloor \epsilon n \rfloor} \binom{n}{k} \leq 2^{H(\epsilon)n},
\]

where \( 0 \leq \epsilon \leq \frac{1}{2} \) and \( H \) is the binary entropy function, \( H(\epsilon) := -\epsilon \log_2(\epsilon) - (1-\epsilon) \log_2(1-\epsilon) \).

Would you still agree to playing this game?

Exercise 7: Random Walks (in-class)

An apple is located at vertex \( A \) of pentagon ABCDE, and a worm is located two vertices away, at C. Every day the worm crawls with equal probability to one of the two adjacent vertices. Thus after one day the worm is at vertex B or D, each with probability \( \frac{1}{2} \). After two days, the worm might be back at C again, because it has no memory of previous positions. When it reaches vertex A, it stops to dine.

(a) What is the mean of the number of days until dinner?

(b) Let \( p \) be the probability that the number of days is 100 or more. What does Markov’s Inequality say about \( p \)?
Exercise 8: Independence of Three Events (in-class)

Give an example of a probability space with three events $A$, $B$, $C$ such that both of the following hold:

- The events are pairwise independent, i.e.
  
  \[
  \Pr[A \cap B] = \Pr[A] \cdot \Pr[B] \\
  \Pr[B \cap C] = \Pr[B] \cdot \Pr[C] \\
  \Pr[A \cap C] = \Pr[A] \cdot \Pr[C]
  \]

- The events are not mutually independent because
  
  \[
  \Pr[A \cap B \cap C] \neq \Pr[A] \cdot \Pr[B] \cdot \Pr[C]
  \]