General rules for solving exercises

- This is a theory course, which means: if an exercise does not explicitly say “you do not need to prove your answer” or “justify intuitively”, then a formal proof is always required.
- All exercises and their solutions, no matter whether they are graded or regular/optional ones, are part of the material relevant for the two exams.
- Some of the exercises are marked as "in-class", which means that we do not expect you to solve them before the exercise session. Instead, your teaching assistant will solve them with you in class.
- You are highly encouraged to solve all other exercises (those not marked as "in-class") on your own and to hand in a writeup of your solutions no later than the due date. If you choose to do so, please write the name of your teaching assistant on the front sheet.

The following exercises will be discussed in the exercise class on September 24, 2014. Please hand in your solutions not later than September 23.

Exercise 1: Multiple Choice Questions

(From the 2011 midterm exam)

(a) For a random permutation \( \pi \) on the keys \([1..n]\), let \( T_\pi \) be the search tree that we obtain from inserting all \( n \) keys, one after the other, in the order given by \( \pi \). Then \( T_\pi \) has the usual search tree distribution if and only if \( \pi \) is distributed uniformly.

[ ] True   [ ] False

Justification: .................................................................

(b) Suppose that a sequence \( \{x_n\}_{n \geq 1} \) with \( x_1 > 0 \) satisfies \( x_n = \sqrt{n} + 2 \sum_{i=1}^{n-1} x_i \) for \( n \geq 2 \).

Then \( x_n = O(2^n) \).

[ ] True   [ ] False

Justification: .................................................................
Exercise 2: A Random Tree? How random?

(Exercise 1.7 from the lecture notes)

Determine the probability of the following tree with 7 nodes.
What is the smallest, what is the largest possible probability of a

tree with 7 nodes?

Exercise 3: Very Deep Nodes

(Exercise 1.9 from the lecture notes)

For $n \in \mathbb{N}$, show that the expected number of nodes of depth $n-1$ in a random search tree for

$n$ keys is $\frac{2^{n-1}}{n!}$. What is the probability that there is a node of depth $n-1$?

Exercise 4: High Trees

(Exercise 1.10 from the lecture notes)

For $n \in \mathbb{N}$, determine the number of trees of height $n-2$ in $B_{[n]}$. (Recall that $B_{[n]}$ is the set of

all search trees with keys $1, 2, \ldots, n$.)

Exercise 5: Solving Recurrences

(Exercise 1.14 from the lecture notes)

Determine closed forms for the following recursively defined series:

1. For $n \in \mathbb{N}$,
   \[
   a_n = \begin{cases} 
   1, & \text{if } n = 1, \\
   1 + \frac{1}{n} \sum_{i=1}^{n-1} a_i, & \text{otherwise}.
   \end{cases}
   \]

2. For $n \in \mathbb{N}$,
   \[
   b_n = \begin{cases} 
   1, & \text{if } n = 1, \\
   2 + \sum_{i=1}^{n-1} b_i, & \text{otherwise}.
   \end{cases}
   \]

3. For $n \in \mathbb{N}$,
   \[
   c_n = \begin{cases} 
   0, & \text{if } n = 0, \\
   n - 1 + \sum_{i=1}^{n} c_{i-1} + c_{n-i}, & \text{otherwise}.
   \end{cases}
   \]

4. For $n \in \mathbb{N}$,
   \[
   d_n = \begin{cases} 
   0, & \text{if } n = 0, \\
   1 + 2 \sum_{i=0}^{n-1} (-1)^{n-i} d_i, & \text{otherwise}.
   \end{cases}
   \]

5. For $n \in \mathbb{N}$,
   \[
   e_n = \begin{cases} 
   1, & \text{if } n = 0, \\
   1 + n e_{n-1}, & \text{otherwise}.
   \end{cases}
   \]
Exercise 6: Descendants of the Smallest Key

Let $S_n$ denote the number of keys that are descendants of the smallest key. For example, in the tree below, $S_n = 5$, because the elements 1, 2, 3, 4, 6 are descendants of 1. Compute $E[S_n]$. 

![Tree Diagram]