General rules for solving exercises

- This is a theory course, which means: if an exercise does not explicitly say “you do not need to prove your answer” or “justify intuitively”, then a formal proof is always required.

- All exercises and their solutions, no matter whether they are graded or regular/optional ones, are part of the material relevant for the two exams.

- Some of the exercises are marked as "in-class", which means that we do not expect you to solve them before the exercise session. Instead, your teaching assistant will solve them with you in class.

- You are highly encouraged to solve all other exercises (those not marked as "in-class") on your own and to hand in a writeup of your solutions no later than the due date. If you choose to do so, please write the name of your teaching assistant on the front sheet.

The following exercises will be discussed in the exercise class on October 8, 2014. Please hand in your solutions not later than October 7.

Exercise 1: Building a Treap (in-class)

Consider the process of inserting the keys \{1, 2, \ldots, n\} into an empty treap in the order \(1, 2, \ldots, n\).

(a) During this process, what is the expected number of changes of the root of the treap? (We also count the very first insertion as a change of the root.)

(b) For a given key \(i\): What is the probability that \(i\) occurs as the right child of the root (after an insertion, i.e., with necessary rotations completed) in the process?

(c) What is the expected number of elements that occur as the left child of the root (after an insertion, i.e., with necessary rotations completed) in the process?

Exercise 2: Comparisons in Quicksort

(Exercise 1.36 from the lecture notes)

\(i, j, n \in \mathbb{N}, \ i < j \leq n\). What is the probability that the randomized procedure quicksort() applied to a set of \(n\) numbers compares the element of rank \(i\) with the element of rank \(j\)?

Exercise 3: Quickselect vs. Random Search Trees

(Exercise 1.39 from the lecture notes)

Let \(X_{k,n}\) be the random variable for the number of comparisons made by quickselect when searching for the element of rank \(k\) in a set of \(n\) numbers. Define a random variable on random search trees with the same distribution.
Exercise 4: Two Closest Numbers

(Exercise 2.1 from the lecture notes)

Suppose you are given a finite set $S \subseteq \mathbb{R}$, $2 \leq |S|$, which is to be preprocessed so that for query $q \in \mathbb{R}$ the answer is 'the' set $(b_1, b_2) \subseteq S$ of the two closest numbers in $S$ (i.e. $\max(|b_1 - q|, |b_2 - q|) \leq \min_{a \in S \setminus \{b_1, b_2\}} |a - q|$). Follow the locus approach for the problem and describe the resulting partition of regions of equal answers (and be aware of the ambiguity issue, i.e. the 'the' has to be taken with caution).

Exercise 5: Locally vs. Globally Convex

(Exercise 2.2 from the lecture notes)

We are given a sequence of points $p_0, p_1, \ldots, p_{n-1}$ in the plane, such that for all $i \in \{0..n-1\}$, the sequence $p_i, p_{i+1}, p_{i+2}$ are vertices of a triangle in counterclockwise order (indices are understood modulo $n$). Is this necessarily the sequence of vertices (in counterclockwise order) of a convex polygon?

Exercise 6: Finding a Key vs. Line Hitting Convex Polygon

(Exercise 2.4 from the lecture notes)

Given a sorted sequence $a_0 < a_1 < \cdots < a_{n-1}$ of $n$ real numbers, we consider the convex polygon $C$ with vertices $(a_i, a_{i+1})$ for $i = 0, \ldots, n-1$. For $k \in \mathbb{R}$, show that the line with equation $y = 2kx - k^2$ intersects $C$ iff $k \in \{a_0, a_1, \ldots, a_{n-1}\}$. Remark: This exercise is supposed to exhibit that deciding whether a line intersects a convex polygon cannot be easier than deciding whether a query key is in a given set of keys.