Linear Programming (for Approximation)

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One source and inspiration for the text below is
Jens Vygen, New approximation algorithms for the TSP,
http://www.or.uni-bonn.de/~vygen/files/optima.pdf

6.7 Traveling Salesman

Given a graph $G = (V, E)$ with costs $c_e \in \mathbb{R}$, $e \in E$, a tour $\tau$ is a spanning cycle (also called Hamiltonian cycle), formally a subset $E_\tau$ of $E$ such that (i) the graph $(V, E_\tau)$ is connected and (ii) every vertex is incident to exactly two edges in $E_\tau$. The cost of tour $\tau$ is defined as $\sum_{e \in E_\tau} c_e$. An optimal tour in $G$ is a tour of minimal cost. The task of computing an optimal tour for a given graph $G$ is often called the traveling salesman problem,\textsuperscript{4} a classical NP-complete problem.

For $S \subseteq V$, let $\delta(S) := \{e \in E : |e \cap S| = 1\}$, sometimes called the boundary of $S$ or the edge set of the cut $(S, V \setminus S)$. For $v \in V$, we write $\delta(v)$ short for $\delta(\{v\})$, the set of edges incident to $v$. With this we can specify

\textsuperscript{3}Thanks also to Thomas Holenstein for discussions on the topic and to Manuel Wettstein for comments after reading a draft version.

\textsuperscript{4}More recently, in the spirit of political correctness, renamed to traveling salesperson problem.
the characteristic vectors of edge sets of tours by the following constraints.

\[
x \in \{0, 1\}^E
\]
\[
\sum_{e \in \delta(v)} x_e = 2, \quad \text{for all } v \in V, \text{ and}
\]
\[
\sum_{e \in \delta(S)} x_e \geq 1, \quad \text{for all } S \subseteq V \text{ with } \emptyset \neq S \neq V. \quad (6.6)
\]

Condition (6.6) says that every nontrivial cut must contain at least one edge, a characterization of connectivity (see exercise below). Note, however, that for the edge set of a tour, every such cut must indeed have at least two edges (follows from the fact that every cut must have even size). That is, we can substitute (6.6) by

\[
\sum_{e \in \delta(S)} x_e \geq 2, \quad \text{for all } S \subseteq V \text{ with } \emptyset \neq S \neq V. \quad (6.7)
\]

While these constraints are equivalent in the integer program, the resulting LP relaxation of the traveling salesmen problem is more constrained and therefore its optimal solution is hopefully closer to the integer solution.

**Subtour LP for graph** \( G = (V, E) \), \( c \in \mathbb{R}^E \)

\[
\min c^T x
\]

subject to

\[
\begin{align*}
\sum_{e \in \delta(v)} x_e &= 2, \quad \text{for all } v \in V, \\
\sum_{e \in \delta(S)} x_e &\geq 2, \quad \text{for all } S \subseteq V \text{ with } \emptyset \neq S \neq V, \text{ and} \\
1 &\geq x_e \geq 0, \quad \text{for all } e \in E.
\end{align*}
\]

This LP is called **Subtour LP** or **Held-Karp relaxation**, although it was first considered by Dantzig, Fulkerson and Johnson in 1954. Recall that an optimal solution \( \tilde{x} \) to this LP is a lower bound for an optimal solution \( x^* \in \{0, 1\}^E \) to the underlying integer program, i.e.

\[
c^T \tilde{x} \leq c^T x^* = \text{OPT}_{\text{tour}}.
\]

with \( \text{OPT}_{\text{tour}} \) the cost of the optimal tour in \( G \) with costs \( c \). (Caveat: Is the LP and IP always feasible? See exercise below.)
6.7. TRAVELING SALESMAN

A graph \( G = (V, E) \) with costs \( c \) satisfies the triangle inequality if \( E = \binom{V}{2} \) and

\[
c_{[u,w]} \leq c_{[u,v]} + c_{[v,w]} \quad \text{for distinct } u, v, w \in V.
\]

(For \( |V| \geq 3 \) it follows that \( c_e \geq 0 \) for all \( e \in E \).) Provided the triangle inequality is satisfied, it can be shown that

\[
c^T \xi \leq c^T \xi^* = OPT_{\text{tour}} \leq \frac{3}{2} c^T \xi.
\]

The 3/2 ratio reminds us of the Christofides approximation for the traveling salesman problem with triangle inequality. We will return to this point later and we will see that there is indeed a connection.

There is a worry looming: The Subtour LP has an exponential number of constraints \( (n + (2^n - 2) + 2m, \text{ for } n := |V| \text{ and } m := |E|) \). So can we solve the LP efficiently? The dimension is \( m \) and given a vector \( x \), we can find a violated cut constraint (6.7), if it exists, via a min-cut algorithm in polynomial time. This provides exactly the type of polynomial separation oracle we need for an efficient employment of the ellipsoid method.

Exercise 6.10 Show that a graph \( G = (V, E) \) is connected iff \( \delta(S) \neq \emptyset \) for all \( S \subseteq V, \emptyset \neq S \neq V \). Recall the definition of “connected.” A graph \( G \) is connected if there is a path between any two vertices in \( G \).

Exercise 6.11 (i) Give a graph \( G = (V, E) \) for which the subtour LP is infeasible. (ii) Give a graph \( G = (V, E) \) where the subtour LP is feasible but there is no feasible integer solution.

Exercise 6.12 Show that the constraints “\( 1 \geq x_e \)” are redundant in the Subtour LP, i.e. every point \( x \in \mathbb{R}^e \) that is feasible w.r.t. all other constraints does satisfy \( 1 \geq x_e \) for all \( e \in E \).

Exercise 6.13 Show that all characteristic vectors \( x \) of tours of a graph \( G = (V, E) \) are vertices of the feasible region of the Subtour LP. (You can use the following characterizations: (i) A point \( p \in \mathbb{R}^m \) is a vertex of a convex polyhedron \( P \subseteq \mathbb{R}^m \) if there exists a hyperplane \( h \) with
Exercise 6.14 Investigate the Subtour LP versus the Loose Subtour LP, where the cut constraint is written with “$\geq 1$” rather than “$\geq 2$”. One concrete question to consider is the following: If $\bar{x}$ is an optimal solution to the Subtour LP and $\bar{x}'$ is an optimal solution to the Loose Subtour LP, how small can $c^T\bar{x}'$ be compared to $c^T\bar{x}$?

6.8 Minimum Spanning Tree

Given a graph $G = (V, E)$ a subgraph $T = (V, E')$, $E' \subseteq E$, is called a spanning tree of $G$ if (i) it is connected and (ii) it has no cycle. Given costs $c_e \in \mathbb{R}$ for $e \in E$, a minimum spanning tree is a spanning tree $T = (V, E')$ with minimal cost $\sum_{e \in E'} c_e$.

It is well-known that there are several equivalent characterizations, e.g.

$T$ is a tree iff (i) it has exactly $n - 1$ edges and (ii) it is connected.

This suggests to express the characteristic vectors of spanning trees by the following constraints.

\[
x \in \{0, 1\}^E \\
\sum_{e \in E} x_e = n - 1, \text{ and} \\
\sum_{e \in \delta(S)} x_e \geq 1, \text{ for all } S \subseteq V \text{ with } \emptyset \neq S \neq V.
\]

We consider the LP relaxation.

**Loose Spanning Tree LP** for graph $G = (V, E)$, $c \in \mathbb{R}^E$

\[
\begin{align*}
\min & \quad c^T x \\
\text{subject to} & \quad \sum_{e \in E} x_e = n - 1 \\
& \quad \sum_{e \in \delta(S)} x_e \geq 1, \text{ for all } S \subseteq V, \emptyset \neq S \neq V, \text{ and} \\
& \quad 1 \geq x_e \geq 0, \text{ for all } e \in E.
\end{align*}
\]
Let us investigate the behavior of the Loose Spanning Tree LP on the graph

\[ G_{k,\ell} := \left( U \cup \{v_0, v_1, \ldots, v_\ell\}, \left( U \cup \{v_0, v_\ell\} \right) \cup \{(v_0, v_1), (v_1, v_2), \ldots, (v_{\ell-1}, v_\ell)\} \right) \]

with |U| = k - 2. That is, this graph has k + \ell - 1 vertices; it consists of a k-clique with a path of length \ell attached to two vertices of the clique. Now define \( c_e := 0 \) for all edges of the k-clique and, for some positive real number \( \gamma \), \( c_e := \gamma \) for all edges of the \ell-path.

**Lemma 6.10**  
(i) For the graph \( G_{k,\ell} \) with costs as described above a minimum spanning tree has cost \((\ell - 1)\gamma\).  
(ii) The corresponding Loose Spanning Tree LP has a value of at most \( \frac{1}{2}\gamma \) if \( \ell = k(k - 3) + 4 \).

**Proof.** Every spanning tree of \( G_{k,\ell} \) must contain at least \( \ell - 1 \) of the edges of the \ell-path, otherwise it cannot be connected. Hence, (i) holds.

For a proof of (ii) let \( x_e := 1/2 \) for the edges of the \ell-path and \( x_e := 1 \) for the edges in the k-clique. This gives \( \sum_{e \in E} x_e = \ell/2 + \binom{k}{2} = (k+\ell-1) - 1 \) for \( \ell = k(k - 3) + 4 \). Therefore the equality of the LP is satisfied, and it is easily seen that the cut constraints are met as well (follows from the fact that there is a spanning cycle where all edges have weight \( x_e \) at least \( \frac{1}{2}\)). The value of \( c^T x \) equals \( \frac{1}{2}\gamma \). \(\square\)
We have seen that the Loose Spanning Tree LP allows a fractional solution that is roughly a factor \( \frac{1}{2} \) smaller than the cost of a minimum spanning tree. A better LP is possible. For that we now switch to the alternative characterization

T is a tree iff (i) it has \( n - 1 \) edges and (ii) it contains no cycle.

A graph has no cycle iff no set of \( k \) vertices induces a graph with \( k \) or more edges. This leads to the following new set of constraints for the spanning tree problem.

**Tight Spanning Tree LP** for graph \( G = (V, E), c \in \mathbb{R}^E \)

\[
\begin{align*}
\text{min} & \quad c^T x \\
\text{subject to} & \quad \sum_{e \in E} x_e = n - 1 \\
& \quad \sum_{e \in E \cap (S, \frac{1}{2})} x_e \leq |S| - 1, \text{ for all } S \subseteq V, \emptyset \neq S \neq V, \text{ and} \\
& \quad 1 \geq x_e \geq 0, \text{ for all } e \in E.
\end{align*}
\]

Here comes a surprise, although we have to leave it without a proof here and refer to the literature for that.

**Theorem 6.11 (Edmonds, 1970)** Every basic feasible solution of the Tight Spanning Tree LP is integral. Therefore, the value of the Tight Spanning Tree LP equals the cost of the minimum spanning tree for every cost vector \( c \).

**The Geometric View and the Spanning Tree Polytope.** We are interested in a certain family of subsets of a ground set, here the family \( T \) of (edge sets of) spanning trees of a graph \( G = (V, E) \). Let \( S_T \subseteq \{0, 1\}^E \) be the set of characteristic vectors of the sets in \( T \). For \( c \in \mathbb{R}^E \), we are interested in\n
\[
\min_{x \in S_T} c^T x \quad \text{(and in an } x^* \in S_T \text{ with } c^T x^* = \min_{x \in S_T} c^T x)\n\]

We now design a linear program \( Ax \leq b \) such that the polyhedron \( \tilde{P} := \{ x \in \mathbb{R}^E : Ax \leq b \} \) specifies our problem in the sense that \( S_T = \tilde{P} \cap \mathbb{Z}^E \). A solution \( \min_{c \in \tilde{P}} c^T x \) to the linear program (with cost vector \( c \)) may obviously deviate from our desired \( \min_{x \in S_T} c^T x \). The better the polyhedron \( \tilde{P} \) "embraces" \( S_T \), the better the LP solution will be.
6.8. MINIMUM SPANNING TREE

There is a “perfect” polyhedron, namely $P^* = P^*_T := \text{conv}(S_T)$. We have $P^* \subseteq [0, 1]^l$, therefore $P^*$ is a polytope\(^5\); it is called the spanning tree polytope of the underlying graph $G$.

It is not hard to show that

$$\min_{x \in P^*} c^T x = \min_{x \in S_T} c^T x .$$

A fundamental theorem in discrete geometry says that every polytope is the intersection of a finite number of halfspaces, that is, there is a linear program $A'x \leq b'$ such that $P^*$ equals $\{ x \in \mathbb{R}^l : A'x \leq b' \}$. Edmonds’ Theorem tell us that the Tight Spanning Tree LP has exactly this property: The set of feasible solutions is $P^*$.

But we just learned that such an LP always exists, so what’s the big deal. The problem is that for our algorithmic intentions (i.e. solving a optimization problem efficiently), the existence of the perfect LP is not enough, we need to get a hand on it, find some concrete description. This description needs not to be small, we do not even need it in some explicit form, but we need some form of separation oracle as we have discussed it for the ellipsoid method.

**Exercise 6.15** Show that the value of the Loose Spanning Tree LP for $G_{k,l}$ with the weights described above and with $l = k(k-3) + 4$ is exactly $\gamma \frac{l}{2}$.

**Exercise 6.16** Show that every feasible point of the Tight Spanning Tree LP is feasible in the Loose Spanning Tree LP – without using Theorem 6.11.

**Exercise 6.17** $m \in \mathbb{N}$. Let $S \subseteq \mathbb{R}^m$ be finite and $P := \text{conv}(S)$. Show that for every $c \in \mathbb{R}^m$ we have

$$\min_{x \in P} c^T x = \min_{x \in S} c^T x .$$

(You may use the Separation Lemma: For $p \in \mathbb{R}^m$ and $C \subseteq \mathbb{R}^m$ a convex set, we have $p \notin C$ iff there exists a closed halfspace $H$ with $p \in H$ and $H \cap C = \emptyset$.)

\(^5\) Follows also from the fact that $S_T$ is finite.
Exercise 6.18 Consider the following linear program, almost the Tight Spanning Tree LP, it seems:

Some LP for graph $G = (V, E), c \in \mathbb{R}^E$

$$\min \ c^T x$$
$$\text{subject to} \quad \sum_{e \in E} x_e = n$$
$$\sum_{e \in \mathcal{E}(S)} x_e \leq |S| - 1, \text{ for all } S \subseteq V, 0 \neq S \neq V, \text{ and}$$
$$1 \geq x_e \geq 0, \text{ for all } e \in E.$$

What are the edge sets corresponding to vectors $x \in \{0, 1\}^E$ feasible in Some LP?

6.9 Back to the Subtour LP

With the euphoria after having seen a “perfect” LP relaxation for the minimum spanning tree problem, we would of course like to know what the situation with the Subtour LP is. More concretely, for $\bar{x}$ an optimal solution to the Subtour LP and for $x^*$ the integral counterpart, do we always have equality $c^T \bar{x} = c^T x^*$. We suspect this not to be true since otherwise we have a polynomial time algorithm for the traveling salesman problem, an NP-complete problem—but why not?

If this equality does not always hold, we are still interested in how big the ratio $\frac{c^T x^*}{c^T \bar{x}}$ can get. This ratio is called the integrality ratio\(^6\) (Edmonds’ Theorem 6.11 implies that the integrality ratio is always 1 for the Tight Spanning Tree LP).

Graphic Metrics. In order to get a hand on interesting cost functions, we make the following definitions. Let $G = (V, E)$ be a connected graph and let $d(u, v)$ be the length of the shortest path between $u$ and $v$. This induces costs $c_{(u, v)} := d(u, v) \in \mathbb{N}$ on the edges of the complete graph $(V, \binom{V}{2})$. These costs always satisfy the triangle inequality. A cost function on the complete graph obtained in this way is called a graphic metric.\(^7\)

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\(^6\)Sometimes also called integrality gap.

\(^7\)The traveling salesman problem for a graphic metric is often called graphic TSP or graph TSP.
6.9. BACK TO THE SUBTOUR LP

Figure 6.2: Graph $G_6$, a “fractional” cycle according to the Subtour LP: If fat edges $e$ get weight $x_e := 1$ and thin edges get weight $x_e := \frac{1}{2}$, then every vertex is incident to edges of overall weight 2, and every cut has weight at least 2.

Lemma 6.12 Let the graphic metric $c$ on $\binom{V}{2}$ be induced by the connected graph $G = (V,E)$. Then the optimal tour of $G' = (V, \binom{V}{2})$ with costs $c$ equals the length of the shortest closed walk\textsuperscript{8} in $G$ visiting all vertices at least once.

Proof. Let $(v_0, v_1, \ldots, v_{\ell-1}, v_\ell = v_0)$ be a closed walk of length $\ell$ in $G$ visiting all vertices, i.e. there are indices $0 = i_1 < i_2 < \ldots < i_n < \ell$ such that $V = \{v_{i_1}, v_{i_2}, \ldots, v_{i_n}\}$. Note that for $1 \leq j < k \leq \ell$, we have $d(v_{i_j}, v_{i_k}) \leq i_k - i_j$ and, therefore, the tour $v_{i_1}, v_{i_2}, \ldots, v_{i_n}$ and back to $v_{i_1}$ is a tour in $G'$ of cost at most $\ell$. In a similar fashion every tour of cost $\ell$ in $G'$ can be turned into a closed walk of length $\ell$ visiting all vertices of $G$. \hfill \qed

Now we consider a specific graph $G_k$ with $n := 3k$ vertices. It consists of three disjoint paths of length $k - 1$ (i.e. $k$ vertices each), plus pairwise edges between the three starting points and also between the three end points of these paths. Let $G'_k$ be the graph with the graphic metric induced by $G_k$.

Lemma 6.13 Let $k \in \mathbb{N}$ and let $n := 3k$ (the number of vertices of $G_k$ and $G'_k$). (i) The Subtour LP on $G'_k$ has a feasible solution with value $n$. (ii) Every closed walk in $G_k$ visiting all vertices has length at least $\frac{2}{3}n - O(1)$ and therefore every tour in $G'_k$ has cost at least $\frac{2}{3}n - O(1)$.

Proof. Note first that for every edge $e$ of $G_k$ we have $c_e = 1$ in $G'_k$. Now set $x_e := \frac{1}{2}$ for the six edges in the two 3-cliques of $G_k$, let $x_e := 1$ for the

\textsuperscript{8}A walk in a graph $G = (V,E)$ is a sequence of (not necessarily distinct) vertices $(v_0, v_1, \ldots, v_\ell)$ with consecutive vertices adjacent in $G$. The walk is closed if $v_\ell = v_0$. The length of the walk is $\ell$, and, for edge costs $c \in \mathbb{R}^E$, the cost of the walk is $\sum_{i=1}^\ell c(v_{i-1}, v_i)$. 
remaining \( n - 3 \) "path"-edges in \( G_k \), and let \( x_e := 0 \) for all other edges in \( G'_k \); hence, \( \sum_e x_e = n \). The resulting vector \( x \) is feasible in the Subtour LP (not too difficult to check, argument omitted here) and \( c^T x = \sum_e x_e = n \), since all edges with \( x_e > 0 \) have \( c_e = 1 \). Thus, (i) is shown.

For (ii) observe that every closed walk visiting all vertices in \( G_k \) has to use all or all but one edge on each of the three building paths of \( G_k \): Skipping two edges disconnects some vertex from the rest and thus the walk cannot visit all vertices. Next consider a set of three edges \( \{e_1, e_2, e_3\} \), with \( e_1 \) from the first path, \( e_2 \) from the second and \( e_3 \) from the third path. Removal of these three edges splits the graph into two parts. It follows that every closed walk has to use the edges in \( \{e_1, e_2, e_3\} \) an even number of times; that is, if all three edges are indeed used at least once, then we have to use them at least 4 times altogether.

Now partition the edges of the three paths in triples, each triple containing one edge from each path. This gives \( k - 1 \) such triples, where at least \((k - 1) - 3\) triples have to be used at least 4 times in a closed walk visiting all vertices. Therefore, the walk must have length at least \( 4(k - 4) = \frac{4}{3} n - 16 \).

We can conclude that the Subtour LP is not perfect and that we have a family of graphs \( G'_k \) with costs \( c \) which exhibit an integrality ratio of roughly \( \frac{4}{3} \):

\[
\frac{c^T x^*}{c^T \bar{x}} \to \frac{4}{3} \quad \text{as the size } n = 3k \text{ of the graphs grows.}
\]

Now, clearly, we wish to know whether the integrality ratio \( \frac{c^T x^*}{c^T \bar{x}} \) has an upper bound. For that we will have to combine the insights we have collected about the Subtour LP, about the Tight Spanning Tree LP, and about the Christofides approximation algorithm for TSP.

**Exercise 6.19** Show that a shortest closed walk visiting all vertices in \( G_k \) has length \( 4k - 2 = \frac{4}{3} n - 2 \).

**Exercise 6.20** Let \( T = (V, E) \) be a tree and let \( G' \) be the complete graph on \( V \) with the graphic metric induced by \( T \). (i) What is the cost of
an optimal tour in $G'$? (ii) What can you say about the value of the Subtour LP value for $G'$? (This is a very unspecified question, so you will have to decide for yourself in which direction to go.)

6.10 Subtour LP versus Tight Spanning Tree LP

We now relate the Subtour LP to the Tight Spanning Tree LP with the goal of deriving an upper bound on the integrality ratio of the Subtour LP.

Lemma 6.14 For a given graph $G = (V, E)$, if $x \in \mathbb{R}^E$ is a feasible solution of the Subtour LP, then $\frac{1}{\sqrt{n}}x$ is a feasible solution of the Tight Spanning Tree LP.

Proof. For $x$ a feasible solution of the Subtour LP we have

$$\sum_{e \in E} x_e = \frac{1}{2} \sum_{v \in V} \sum_{e \in \delta(v)} x_e = \frac{1}{2} 2n = n$$

and therefore $\sum_{e \in E} \frac{n-1}{n} x_e = n - 1$ and the first constraint of the Tight Spanning Tree LP is satisfied for $\frac{n-1}{n} x$.

Next, for $S \subseteq V$, $\emptyset \neq S \neq V$, we have

$$\sum_{e \in E \cap \left\{ \frac{1}{2} \right\}} x_e = \frac{1}{2} \left( \sum_{v \in S} \sum_{e \in \delta(v)} x_e \right) \geq \frac{1}{2} \left( 2|S| - 2 \right) = |S| - 1 .$$

We have shown that also the inequalities of the Tight Spanning Tree LP are satisfied. (Even $1 \geq \frac{n-1}{n} x_e \geq 0$ follows from $1 \geq x_e \geq 0$.)

Corollary 6.15 Given a graph $G$, if $\bar{x}$ is an optimal solution of the Subtour LP and $\text{OPT}_{\text{mst}}$ is the cost of the minimum spanning tree, then $c^T \bar{x} \geq \frac{n-1}{n} \text{OPT}_{\text{mst}}$.

Proof. We know that $\frac{n-1}{n} \bar{x}$ is a feasible solution of the Tight Spanning Tree LP, and therefore $\frac{n-1}{n} c^T \bar{x}$ is at least the value of the Tight Spanning Tree LP, which is attained by a basic feasible solution, which is known to be integral (by Theorem 6.11) and therefore the value equals $\text{OPT}_{\text{mst}}$. □
Recall that, under the assumption of the triangle inequality, we have

$$\text{OPT}_{\text{tour}} \leq 2 \cdot \text{OPT}_{\text{mat}}$$  \hspace{1cm} (6.8)

(We can turn the closed walk that uses every edge of a minimum spanning tree twice into a tour by skipping vertices already visited; the triangle inequality ensures that the resulting tour has cost at most the cost of the initial closed walk, which is twice the cost of the minimum spanning tree.)

**Corollary 6.16** For a graph $G$ whose costs $c$ satisfy the triangle inequality the integrality ratio of the Subtour LP is at most 2.

**Proof.** For $\bar{x}$ an optimal solution to the Subtour LP and $x^*$ an optimal integral solution, we have

$$c^T \bar{x} \geq \frac{n}{n-1} \text{OPT}_{\text{mat}} \geq \text{OPT}_{\text{mat}} \geq \frac{1}{2} \text{OPT}_{\text{tour}} = \frac{1}{2} c^T x^*;$$

the first inequality is from Corollary 6.15, the second inequality uses that $\text{OPT}_{\text{mat}}$ is non-negative (because of the triangle inequality), the third inequality is from (6.8) above, and the last equality is the basic property of the Subtour LP. Hence, $\frac{c^T x^*}{c^T \bar{x}} \leq 2$.

This bound of 2 allows improvement along the familiar Christofides approximation, which generates a tour at most $3/2$ times the cost of the optimal tour as follows: (1) Choose a minimum spanning tree (of cost $\text{OPT}_{\text{mat}}$). (2) For $U$ the set of vertices of odd degree in this tree, compute the minimum weight matching (of cost $\text{OPT}_{\text{match}(U)}$) covering all vertices in $U$ (this set is conveniently of even size). (3) The union of the minimum spanning tree and the matching is Eulerian (i.e. is connected and all vertices have even degree, if edges both in tree and matching are counted twice), thus a closed walk of cost $\text{OPT}_{\text{mat}} + \text{OPT}_{\text{match}(U)}$ visiting all vertices exists. (4) Such a walk can be turned into a tour of at most this cost (exploiting the triangle inequality).

We can can conclude that

$$\text{OPT}_{\text{tour}} \leq \text{OPT}_{\text{mat}} + \text{OPT}_{\text{match}(U)}$$

Besides $\text{OPT}_{\text{mat}} \leq c^T \bar{x}$ one can also show $\text{OPT}_{\text{match}(U)} \leq \frac{1}{2} c^T \bar{x}$ (proof omitted here). With these facts in our hands, we can conclude:
Theorem 6.17 If \( G \) is a complete graph with costs satisfying the triangle inequality and if \( \bar{x} \) is an optimal solution to the Subtour LP for \( G \), then
\[
c^T \bar{x} \leq \text{OPT}_{\text{tour}} \leq \frac{3}{2}c^T \bar{x},
\]
(for \( \text{OPT}_{\text{tour}} \) the cost of an optimal tour in \( G \)).

For all \( \varepsilon > 0 \) there exists a graph with costs satisfying the triangle inequality such that \( \text{OPT}_{\text{tour}} \geq (\frac{4}{3} - \varepsilon)c^T \bar{x} \).

The situation between \( \frac{4}{3} \) and \( \frac{3}{2} \) has been open for a long time – and still is, to the best of my knowledge. Also, Christofides’ polynomial \( 3/2 \)-approximation is still the best known for triangle inequality TSP. However, there was some progress on graphic TSP (with a graphic metric), where Mömke and Svensson, 2011, showed a 1.461-approximation, and later a 1.4-approximation by Sebő and Vygen, 2012.

For open problems in this context see Vygen’s survey [http://www.or.uni-bonn.de/~vygen/files/optima.pdf](http://www.or.uni-bonn.de/~vygen/files/optima.pdf), Section 8 (page 22).