Chapter 1
Some Basic Geometry

This chapter reviews some material we will need during the course and tries to get you acquainted with some unusual phenomena of high-dimensional objects.

1.1 Euclidean Space

The $d$-dimensional Euclidean space $\mathbb{E}^d$ is the $d$-dimensional vector space over the real numbers $\mathbb{R}$, equipped with the scalar product

$$p \cdot q := \sum_{i=1}^{d} p_i q_i, \quad p, q \in \mathbb{E}^d.$$  

Members of $\mathbb{E}^d$, for example $p = (p_1, \ldots, p_d)$ and $q = (q_1, \ldots, q_d)$ are called points or vectors, depending on how we think about them. Typically, a point denotes some absolute location in space (relative to the origin), while a vector stands for the difference $p - q$ of two points. Two special points we frequently need are $0 := (0, \ldots, 0)$ (the origin) and $1 := (1, \ldots, 1)$, with dimensions clear from the context.

The scalar product defines the Euclidean norm of a point (or the length of a vector) via

$$\|p\| := \sqrt{p \cdot p}, \quad p \in \mathbb{E}^d.$$  

It holds that

$$\|t p\| = |t| \|p\|, \quad p \in \mathbb{E}^d, t \in \mathbb{R}.$$  

A basic and important fact is the

**Triangle Inequality.** $\|q - p\| \leq \|q - r\| + \|r - p\|, \quad p, q, r \in \mathbb{E}^d.$

It says that in any triangle, any of the three sides is at most as long as the sum of the two other sides, see Figure 1.1.

A simple but very useful fact is the
Cauchy-Schwarz inequality.

\[ |p \cdot q| \leq \|p\| \|q\|, \quad p, q \in \mathbb{R}^d. \]

The scalar product also defines angles, according to

\[ \cos(\alpha) = \frac{(q - p) \cdot (r - p)}{\|q - p\| \|r - p\|}, \]

see Figure 1.1. We frequently need the

**Cosine Theorem.** \[ \|q - r\|^2 = \|r - p\|^2 + \|q - p\|^2 - 2\|r - p\| \|q - p\| \cos(\alpha). \]

For \( \alpha = 90^\circ \), this is Pythagoras’s Theorem.

### 1.2 Hyperplanes

Any \((d + 1)\)-tuple \((h_1, \ldots, h_d, h_{d+1}) \in \mathbb{R}^{d+1}\) with \((h_1, \ldots, h_d) \neq 0\) defines a hyperplane

\[ h = \{ x \in \mathbb{R}^d \mid \sum_{i=1}^{d} h_i x_i = h_{d+1} \}. \tag{1.1} \]

For \( d = 2 \), hyperplanes are lines (see Figure 1.2), and for \( d = 3 \), we get planes. Note that \( h \) is invariant under scaling its defining \((d + 1)\)-tuple by any nonzero constant.

The vector \( \overrightarrow{h} = (h_1, \ldots, h_d) \in \mathbb{R}^d \) is the so-called normal vector of \( h \). It is orthogonal to the hyperplane in the sense that

\[ \overrightarrow{h} \cdot (p - q) = 0, \quad p, q \in h, \]

a fact that immediately follows from (1.1). It is not hard to prove (basic calculus) that the distance of \( h \) to the origin is \( |h_{d+1}|/\|\overrightarrow{h}\| \), attained by the unique point \((h_{d+1}/\|\overrightarrow{h}\|)\overrightarrow{h}) \).

Any hyperplane \( h \) comes with two halfspaces

\[ h^+ := \{ x \in \mathbb{R}^d \mid \sum_{i=1}^{d} h_i x_i \geq h_{d+1} \}, \]

\[ h^- := \{ x \in \mathbb{R}^d \mid \sum_{i=1}^{d} h_i x_i \leq h_{d+1} \}. \]
In the following, we make the convention that \( h_{d+1} \geq 0 \) (to achieve this, we can scale by \(-1\), if necessary). In this case, \( h^- \) contains the origin. Note that \( h^+ \) and \( h^- \) are well-defined only if \( h_{d+1} \neq 0 \), equivalently, if \( h \) does not contain the origin.

In a slight abuse of notation, we identify the hyperplane \( h \) with its defining \((d+1)\)-tuple \((h_1, \ldots, h_{d+1})\), i.e. we write \( h = (h_1, \ldots, h_{d+1}) \).

**Non-vertical hyperplanes.** Hyperplanes \( h \) with \( h_d \neq 0 \) are called *non-vertical* and have an alternative definition in terms of only \( d \) parameters. Any \( d \)-tuple \((g_1, \ldots, g_d)\) defines a non-vertical hyperplane

\[
g = \{ x \in \mathbb{E}^d \mid x_d = \sum_{i=1}^{d-1} g_i x_i + g_d \}. \tag{1.2}
\]

In this form, the line from Figure 1.2 has the equation

\[
x_2 = -\frac{1}{2} x_1 + 3.
\]

The non-vertical hyperplane \( g \) defines the two halfspaces

\[
g^+ := \{ x \in \mathbb{E}^d \mid x_d \geq \sum_{i=1}^{d-1} g_i x_i + g_d \},
\]

\[
g^- := \{ x \in \mathbb{E}^d \mid x_d \leq \sum_{i=1}^{d-1} g_i x_i + g_d \}.
\]

\( g^+ \) is the halfspace *above* \( g \), while \( g^- \) is *below* \( g \). As above, we identify the non-vertical hyperplane \( g \) with its defining \( d \)-tuple \((g_1, \ldots, g_d)\) by writing \( g = (g_1, \ldots, g_d) \).
1.3 Duality

Points and hyperplanes behave in the same way. Even if it is not clear what this exactly means, the statement may appear surprising at first sight. Here are two duality transforms that map points to hyperplanes and vice versa, in such a way that relative positions of points w.r.t. hyperplanes are preserved.

**The origin-avoiding case.** For \( p = (p_1, \ldots, p_d) \in \mathbb{E}^d \setminus \{0\} \), the origin-avoiding hyperplane

\[
p^* = (p_1, \ldots, p_d, 1) = \{ x \in \mathbb{E}^d \mid \sum_{i=1}^{d} p_i x_i = 1 \}
\]

(1.3)

is called the hyperplane dual to \( p \). Vive versa, for an origin-avoiding hyperplane \( h = (h_1, \ldots, h_{d+1}) \), the point

\[
h^* = \left( \frac{h_1}{h_{d+1}}, \ldots, \frac{h_d}{h_{d+1}} \right) \in \mathbb{E}^d \setminus \{0\}
\]

(1.4)

is called the point dual to \( h \). We get \((p^*)^* = p\) and \((h^*)^* = h\) (modulo scaling of coordinates by a positive multiple), so this duality transform is an involution (a mapping satisfying \( f(f(x)) = x \) for all \( x \)).

It follows from the above facts about hyperplanes that \( p^* \) is orthogonal to \( p \) and has distance \( 1/\|p\| \) from the origin. Thus, points close to the origin are mapped to hyperplanes far away, and vice versa. \( p \) is actually on \( p^* \) if and only if \( \|p\| = 1 \), i.e. if \( p \) is on the so-called unit sphere, see Figure 1.3.

![Figure 1.3: Duality in the origin-avoiding case](image)

The important fact about the duality transform is that relative positions of points w.r.t. hyperplanes are maintained.

**Lemma 1.3.1** For all points \( p \neq 0 \) and all origin-avoiding hyperplanes \( h \), we have

\[
p \in \left\{ \begin{array}{l} h^+ \\ h^- \\ h \end{array} \right\} \iff h^* \in \left\{ \begin{array}{l} (p^*)^+ \\ (p^*)^- \\ p^* \end{array} \right\}.
\]
Proof. Really boring, but still useful in order to see what happens (or rather, that nothing happens). Let’s look at $h^+$, the other cases are the same.

\[ p \in h^+ : \iff \sum_{i=1}^{d} h_ip_i \geq h_{d+1} \iff \sum_{i=1}^{d} p_i \frac{h_i}{h_{d+1}} \geq 1 \iff h^* \in (p^*)^+. \]

The non-vertical case. The previous duality has two kinds of singularities: it does not work for the point $p = 0$, and it does not work for hyperplanes containing 0. The following duality has only one kind of singularity: it does not work for vertical hyperplanes, but it works for all points.

For $p = (p_1, \ldots, p_d) \in \mathbb{E}^d$, the non-vertical hyperplane

\[ p^* = (2p_1, \ldots, 2p_{d-1}, -p_d) = \{x \in \mathbb{E}^d \mid x_d = \sum_{i=1}^{d-1} 2p_ix_i - p_d \} \quad (1.5) \]

is called the hyperplane dual to $p$.\(^1\) Vice versa, given a non-vertical hyperplane $g = (g_1, \ldots, g_d)$, the point

\[ g^* = \left( \frac{1}{2} g_1, \ldots, \frac{1}{2} g_{d-1}, -g_d \right) \quad (1.6) \]

is called the point dual to $g$.

Here is the analog of Lemma 1.3.1.

Lemma 1.3.2 For all points $p$ and all non-vertical hyperplanes $g$, we have

\[ p \in \left\{ \begin{array}{c} g^+ \\ g^- \\ g \end{array} \right\} \iff g^* \in \left\{ \begin{array}{c} (p^*)^+ \\ (p^*)^- \\ p^* \end{array} \right\}. \]

Proof. Again, this is really easy (and we only do the $g^+$-case).

\[ p \in g^+ : \iff p_d \geq \sum_{i=1}^{d-1} g_ip_i + g_d \]

\[ \iff -g_d \geq \sum_{i=1}^{d-1} 2p_i \frac{1}{2} g_i - p_d \]

\[ \iff g^* \in (p^*)^+ \]

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\(^1\)We could use another symbol to distinguish this from the previous duality, but since we never mix both dualities, it will always be clear to which one we refer.
It turns out that this duality has a geometric interpretation involving the unit paraboloid instead of the unit sphere [3]. Which of the two is more practical depends on the application.

Duality allows us to translate statements about hyperplanes into statements about points, and vice versa. Sometimes, the statement is easier to understand after such a translation. Exercise 2 gives a nontrivial example. Here is one very easy translation in the non-vertical case. In the origin-avoiding case, the essence is the same, but the precise statement is slightly different (Exercise 3).

**Observation 1.3.3** Let \( p, q, r \) be points in \( \mathbb{E}^2 \). The following statements are equivalent, see Figure 1.4.

(i) The points \( p, q, r \) are collinear (lie on the common line \( \ell \)).

(ii) The lines \( p^*, q^*, r^* \) are concurrent (go through the common point \( \ell^* \)), or are parallel to each other, if \( \ell \) is vertical.

![Figure 1.4: Duality: collinear points translate to concurrent lines (top) or parallel lines (bottom)](image)

**1.4 Convex Sets**

A set \( K \subseteq \mathbb{E}^d \) is called convex if for all \( p, q \in K \) and for all \( \lambda \in [0, 1] \), we also have

\[
(1 - \lambda)p + \lambda q \in K.
\]

Geometrically, this means that for any two points in \( K \), the connecting line segment is completely in \( K \), see Figure 1.5.
It immediately follows that the intersection of convex sets is convex. Convex sets are “nice” sets in many respects, and we often consider the convex hull of a set.

**Lemma 1.4.1** Let $X, K \subseteq \mathbb{E}^d$. The following statements are equivalent, and if $K$ satisfies any of them, $K$ is the convex hull $\text{conv}(X)$ of $X$.

(i) $K$ is the intersection of all convex sets containing $X$,

$$K = \bigcap_{C \supseteq X : C \text{ convex}} C.$$

(ii) $K$ is the intersection of all halfspaces containing $X$,

$$K = \bigcap_{H \supseteq X : H \text{ halfspace}} H.$$

(iii) $K$ is the set of all convex combinations of elements of $X$,

$$K = \left\{ \sum_{x \in S} \lambda_x x \mid S \subseteq X \text{ finite}, \sum_{x \in S} \lambda_x = 1, \forall x \in S : \lambda_x \geq 0 \right\}.$$

Of particular interest for us are convex hulls of point clouds (finite point sets), see Figure 1.6 for an illustration in $\mathbb{E}^2$.

Here is one very important statement about convex sets.

**Helly’s Theorem.** Let $C_1, \ldots, C_n$ be $n \geq d + 1$ convex sets in $\mathbb{E}^d$. If every $d+1$ of the sets have a non-empty common intersection, the common intersection of all sets is nonempty.

For a proof, see Edelsbrunner [3], and for an application, see Exercise 2.
1.5 Balls and Boxes

Here are the most fundamental convex sets in $\mathbb{E}^d$ (see also Exercise 4).

**Definition 1.5.1** Fix $d \in \mathbb{N}, d \geq 1$.

(i) Let $\underline{b} = (b_1, \ldots, b_d) \in \mathbb{R}^d$ and $\overline{b} = (\overline{b}_1, \ldots, \overline{b}_d) \in \mathbb{R}^d$ be two $d$-tuples such that $b_i < \overline{b}_i$ for $i = 1, \ldots, d$. The box $Q_d(\underline{b}, \overline{b})$ is the $d$-fold Cartesian product

$$Q_d(\underline{b}, \overline{b}) := \prod_{i=1}^{d} [b_i, \overline{b}_i] \subseteq \mathbb{E}^d.$$ 

(ii) $Q_d := Q_d(0, 1)$ is the unit box, see Figure 1.7 (left).

(iii) Let $c \in \mathbb{E}^d, \rho \in \mathbb{R}^+$. The ball $B_d(c, \rho)$ is the set

$$B_d(c, \rho) = \{x \in \mathbb{E}^d \mid \|x - c\| \leq \rho\}.$$ 

(iv) $B_d := B_d(0, 1)$ is the unit ball, see Figure 1.7 (right).

While we have a good intuition concerning balls and boxes in dimensions 2 and 3, this intuition does not capture the behavior in higher dimensions. Let us discuss a few counterintuitive phenomena.

**Diameter.** The diameter of a compact$^2$ set $X \subseteq \mathbb{E}^d$ is defined as

$$\text{diam}(X) = \max_{x,y \in X} \|x - y\|.$$ 

What can we say about the diameters of balls and boxes?

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$^2$ a set that is closed and bounded
Lemma 1.5.2  For $d \in \mathbb{N}, d \geq 1$,

(i) $\text{diam}(Q_d) = \sqrt{d}$, and

(ii) $\text{diam}(B_d) = 2$.

Proof. This is not difficult, but it is instructive to derive it using the material we have. For $x, y \in Q_d$, we have $|x_i - y_i| \leq 1$ for $i = 1, \ldots, d$, from which

$$
\|x - y\|^2 = (x - y) \cdot (x - y) = \sum_{i=1}^{d} (x_i - y_i)^2 \leq d
$$

follows, with equality for $x = 0, y = 1$. This gives (i). For (ii), we consider $x, y \in B_d$ and use the triangle inequality to obtain

$$
\|x - y\| \leq \|x - 0\| + \|0 - y\| = \|x\| + \|y\| \leq 2,
$$

with equality for $x = (1, 0, \ldots, 0), y = (-1, 0, \ldots, 0)$. This is (ii).

The counterintuitive phenomenon is that the unit box contains points which are arbitrarily far apart, if $d$ only gets large enough. For example, if our unit of measurement is cm (meaning that the unit box has side length 1cm), we find that $Q_{10,000}$ has two opposite corners which are 1m apart; for $Q_{10^{10}}$, the diameter is already 1km.

Volume. The *volume* of a compact set $X \subseteq \mathbb{E}^d$ is defined as

$$
\text{vol}(X) = \int_{\mathbb{E}^d} \chi_X(x) dx,
$$

where $\chi_X$ is the *characteristic function* of $X$,

$$
\chi_X(x) = \begin{cases} 
1, & \text{if } x \in X, \\
0, & \text{otherwise}.
\end{cases}
$$
Lemma 1.5.3 Let $d \in \mathbb{N}, d \geq 1$.

(i) $\text{vol}(Q_d) = 1$, and

(ii) $\text{vol}(B_d) = \pi^{d/2}/\Gamma\left(\frac{d}{2} + 1\right)$, where $\Gamma$ is the Gamma function. In particular,

$$
\Gamma\left(\frac{d}{2} + 1\right) = \begin{cases} 
\frac{d}{2}!, & \text{if } d \text{ is even}, \\
\pi^{(d-1)/2} \prod_{m=0}^{d-1} \left(m + \frac{1}{2}\right), & \text{if } d \text{ is odd}.
\end{cases}
$$

We skip the proof, because it takes us too far away from our actual topic; here is just the rough idea for (ii): Cavalieri’s principle says that the volume of a compact set in $\mathbb{E}^d$ can be calculated by integrating over the $(d-1)$-dimensional volumes of its slices, obtained by cutting the set orthogonal to some fixed direction. In case of a ball, these slices are balls again, so we can use induction to reduce the problem in $\mathbb{E}^d$ to the problem in $\mathbb{E}^{d-1}$.

Let us discuss the counterintuitive implication of Lemma 1.5.3. The intuition tells us that the unit ball is larger than the unit box, and for $d = 2$, Figure 1.7 clearly confirms this. $B_2$ is larger than $Q_2$ by a factor of $\pi$ (the volume of $B_2$). You might recall (or derive from the lemma) that

$$\text{vol}(B_3) = \frac{4}{3}\pi,$$

meaning that $B_3$ is larger than $Q_3$ by a factor of more than four. Next we get

$$\text{vol}(B_4) \approx 4.93, \quad \text{vol}(B_5) \approx 5.26,$$

so $\text{vol}(B_d)/\text{vol}(Q_d)$ seems to grow with $d$. Calculating

$$\text{vol}(B_6) \approx 5.17$$

makes us sceptical, though, and once we get to

$$\text{vol}(B_{13}) \approx 0.91,$$

we have to admit that the unit ball in dimension 13 is in fact smaller than the unit box. From this point on, the ball volume rapidly decreases (Table 1.1), and in the limit, it even vanishes:

$$\lim_{d \to \infty} \text{vol}(B_d) = 0,$$

because $\Gamma(d/2 + 1)$ grows faster than $\pi^{d/2}$. 

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Table 1.1: Unit ball volumes

<table>
<thead>
<tr>
<th>d</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>⋯</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{vol}(B_d)$</td>
<td>0.91</td>
<td>0.6</td>
<td>0.38</td>
<td>0.24</td>
<td>0.14</td>
<td>⋯</td>
<td>0.026</td>
</tr>
</tbody>
</table>

Figure 1.8: The giant lawn mower covers a stripe of width $h$

Area. Suppose the surface of the earth is completely covered with grass, and your task is to mow it. You have a giant lawn mower able to mow a stripe that spans a spherical angle of $\alpha$, say (where $\alpha$ is small in order not to make your task too easy). What percentage of the grass have you mowed after you have gone around the equator once? See Figure 1.8 (left) for an illustration of the situation.

Obviously, this is a question about the surface area of (parts of) a threedimensional ball. We will not get into the business of formally defining area here; luckily, others have done the work for us, and the lawn mower question is easy to solve using a known fact. The area covered by pushing the lawn mower around the equator is

$$2\pi \rho h,$$

with $\rho$ the radius of the ball (in our case, $\rho \approx 6,378\text{km}$) and $h$ the width of the stripe. Interestingly, this area does not depend on the stripe being centered around the equator—any stripe of width $h$ has area $2\pi \rho h$, see Figure 1.8 (right). For $h = 2\rho$, the stripe covers the whole surface, and the area is $4\pi \rho^2$, the known formula for the surface area of $B_3(0, \rho)$.

Because the stripe spans spherical angle $\alpha$, we get $h = 2\rho \sin(\alpha/2)$, meaning that the fraction of the earth’s surface you have mowed is

$$\frac{2\pi \rho h}{4\pi \rho^2} = \frac{h}{2\rho} = \sin(\alpha/2).$$

If $\alpha = 10^\circ$, for example (a pretty big mower, the stripe is more than 1,000km wide), the fraction covered is 8.7%.

The counterintuitive phenomenon is that your task would be much simpler if the earth were of higher dimension. For sufficiently large dimension, one round with your
10°-mower (or any α-mower, for fixed α) covers 90% (or any desired percentage) of the surface. This means, the surface area of $B_d$ is concentrated around the equator for large $d$. Not only that: by symmetry of $B_d$, the surface area is concentrated around any equator. Figure 1.9 shows the (width of the) stripe around the equator that contains 90% of the area, for three values of $d$, see Matoušek’s book [7]).

![Figure 1.9: Stripe around the equator containing 90% of the area](image)

Without seeing the connection yet, you already know a similar phenomenon involving the unit box $Q_d$. Let the “equator” of $Q_d$ be the set

$$\{ x \in Q_d \mid \sum_{i=1}^{d} x_i = \frac{d}{2} \},$$

see Figure 1.10 (left) for a picture in dimension 3.

![Figure 1.10: The “equator” of the unit box (left); symbolic drawing of a stripe of width $h$ around the equator that contains 96.3% of all box corners (right)](image)

Note that the equator is the intersection of $Q_d$ with a hyperplane. Motivated by Figure 1.9, we plan to prove now that the stripe around the equator containing 90% of the $2^d$ box corners becomes thinner and thinner (compared to the diameter of the box), as $d$ grows. It turns out that this is nothing else than the well-known

**Chernoff Bound.** Let $X_1, \ldots, X_d$ be independent random variables with $\text{prob}(X_i = 0) = \text{prob}(X_i = 1) = 1/2$ for all $i$, and let $X = X_1 + \cdots + X_d$. For
all \( \delta > 0 \), we have
\[
\text{prob} \left( \left| X - \frac{d}{2} \right| > \frac{\delta d}{2} \right) < 2 \exp \left( -\delta^2 \frac{d}{4} \right).
\]

This follows from the **lower tail bound** for independent Poisson trials (see for example [8, Theorem 4.2]), together with the fact that upper and lower tails have the same distribution in our symmetric case.

Setting \( \delta = 4/\sqrt{d} \), for example, yields
\[
\text{prob} \left( \left| X - \frac{d}{2} \right| > 2\sqrt{d} \right) < 2 \exp(-4) \approx 0.037.
\]

Because \( X \) is the sum of coordinates of a randomly chosen unit box corner, it follows that a fraction of no more than 3.7% of all box corners is outside the stripe
\[
\{ x \in \mathbb{E}^d \mid \frac{d}{2} - 2\sqrt{d} \leq \sum_{i=1}^{d} x_i \leq \frac{d}{2} + 2\sqrt{d} \}
\]
around the equator. It follows from our earlier material on hyperplanes (calculation of distance to the origin) that the width of this stripe is
\[
\frac{d/2 + 2\sqrt{d}}{\sqrt{d}} - \frac{d/2 - 2\sqrt{d}}{\sqrt{d}} = 4,
\]
a constant! As \( d \) gets larger, the stripe therefore becomes thinner and thinner compared to the stripe
\[
\{ x \in \mathbb{E}^d \mid 0 \leq \sum_{i=1}^{d} x_i \leq d \}
\]
of width \( \sqrt{d} \) containing the whole unit box, see Figure 1.10 (right) for a symbolic picture.

### 1.6 Exercises

**Exercise 1** A finite point set \( P \subseteq \mathbb{E}^d \) is called affinely independent if the two equations
\[
\sum_{p \in P} \lambda_p p = 0, \quad \sum_{p \in P} \lambda_p = 0
\]
imply \( \lambda_p = 0 \) for all \( p \in P \). Prove that if \( P \) is an affinely independent point set with \( |P| = d \), then there exists a unique hyperplane containing all points in \( P \). (This generalizes the statement that there is a unique line through any two distinct points.)
Exercise 2 Let $S$ be a set of vertical line segments in $\mathbb{E}^2$, see Figure 1.11. Prove the following statement: if for every three of the line segments, there is a line that intersects all three segments, then there is a line that intersects all segments.

Hint. Use the duality transform (non-vertical case) and Helly’s Theorem. For this, you need to understand the following: (i) what is the set of lines dual to the set of points on a (vertical) segment? (ii) if a line intersects the segment, what can we say about the point dual to this line?

Exercise 3 State and prove the analog to Observation 1.3.3 for the origin-avoiding case.

Exercise 4 Prove that all boxes $Q_d(b, b)$ and all balls $B(c, \rho)$ are convex sets.

Exercise 5 In order to generate a random point $p$ in $B_d$, we could proceed as follows: first generate a random point $p$ in $Q_d(-1, 1)$ (this is easy, because it can be done coordinatewise); if $p \in B_d$, we are done, and if not, we repeat the choice of $p$ until $p \in B_d$ holds. Explain why this is not necessarily a good idea. For $d = 20$, what is the expected number of trials necessary before the event ‘$p \in B_d$’ happens?

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3 a line segment is the convex hull of a set of two points