Exercise 1

A finite point set $P \subseteq \mathbb{E}^d$ is called \textit{affinely independent} if the two equations

\[
\sum_{p \in P} \lambda_p p = 0 \quad \text{and} \quad \sum_{p \in P} \lambda_p = 0
\]

imply $\lambda_p = 0$ for all $p \in P$. Prove that if $P \subseteq \mathbb{E}^d$ is an affinely independent point set with $|P| = d$, then there exists a unique hyperplane containing all points in $P$. (This generalizes the statement that there is a unique line through any two distinct points.)

Exercise 2

Let $S$ be a set of vertical line segments\(^1\) in $\mathbb{E}^2$, see Figure below.

![A set of vertical line segments in $\mathbb{E}^2$.](image)

Prove the following statement: if for every three of the line segments, there is a line that intersects all three segments, then there is a line that intersects all segments.

\textbf{Hint:} Use the duality transform (non-vertical case) and Helly’s Theorem. For this, you need to understand the following:

(i) what is the set of lines dual to the set of points on a (vertical) segment?

(ii) if a line intersects the segment, what can we say about the point dual to this line?

Exercise 3

State and prove the analog to Observation 1.3.3 (from the Lecture Notes) for the origin-avoiding case.

Exercise 4

Prove that all boxes $Q_d((a, b))$ and all balls $B_d(c, r)$ are convex sets.

Exercise 5

In order to generate a random point $p$ in $B_d$ we could proceed as follows: first generate a random point $p$ in $Q_d(-1, 1)$ (this is easy, because it can be done coordinatewise); if $p \in B_d$ we are done, and if not, we repeat the choice of $p$ until $p \in B_d$ holds.

Explain why this is not necessarily a good idea. For $d = 20$, what is the expected number of trials necessary before the event ‘$p \in B_d$’ happens?

\(^1\)a line segment is the convex hull of a set of two points