Approximate Methods in Geometry

Exercise Set 4

Course Webpage: http://www.ti.inf.ethz.ch/ew/courses/ApproxGeom05/

Due date: 03.05.2005

Exercise 1

Prove Theorem 3.2.1(i) from the lecture notes:

Let \( P = \{p_1, \ldots, p_n\} \subseteq \mathbb{E}^d \), and let \( Q \) be the \((d \times n)\)-matrix whose columns are the points of \( P \), i.e.

\[
Q = \begin{pmatrix}
p_{11} & \cdots & p_{n1} \\
p_{21} & \cdots & p_{n2} \\
\vdots & \ddots & \vdots \\
p_{1d} & \cdots & p_{nd}
\end{pmatrix}.
\]

Prove that \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) defined by

\[
f(x) = x^T Q^T Q x - \sum_{i=1}^{n} p_i^T p_i x_i, \quad x = (x_1, \ldots, x_n),
\]

is a convex function.

Exercise 2

Use the arguments in the proof of Theorem 3.2.1(ii) to prove Lemma 2.2.2.

Lemma 2.2.2: Let \( P \subseteq \mathbb{E}^d \), \( |P| = n \), and let \( B(P) = B_d(c_P, R_P) \) be the smallest enclosing ball of \( P \). Then for any \( c \in \mathbb{E}^d \) there is a point \( p \in P \) such that

(i) \( \|p - c_P\| = R_P \), and

(ii) \( \|p - c\|^2 \geq \|c - c_P\|^2 + \|p - c_P\|^2 \).

Exercise 3

Formulate the polytope distance problem in the form of a quadratic program (QP).

Can you do it in such a way that the formulation only involves scalar products of points from \( \mathbb{E}^d \)?

Exercise 4

The moment curve in \( \mathbb{E}^d \) is the point set

\[
M_d = \{x \in \mathbb{E}^d \mid x = (t, t^2, \ldots, t^d), \quad t \in \mathbb{R}\}.
\]

Let \( P \subseteq M_d \), \( |P| = n \). Prove that the radius \( R_P \) of \( B(P) \) can be computed in time independent from \( d \).