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Approximate Methods in Geometry *SoSe 2006*

Exercise Set 1

Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/ApproxGeom06/`

Due date: April 11, 2006

Since there was no exercise session today, please contact `uli@inf.ethz.ch` or `sceva@inf.ethz.ch` if you have questions or comments regarding this exercise sheet. (And please notice that there a fourth exercise on the back.)

Exercise 1

(A Warm-Up.) Check that the following sets are convex: (a) the *unit ball* $B = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$; (b) any halfspace; (c) the soldid unit cube $Q = \{x \in \mathbb{R}^d : 0 \leq x_i \leq 1 \text{ for all } i\}$.

Which of these sets are compact?

Exercise 2

A hyperplane $h = \{x \in \mathbb{R}^d : \langle a, x \rangle = \alpha\}$ is called *nonvertical* if $a_d \neq 0$, where $a = (a_1, \ldots, a_d)$ is the normal vector. Observe that we can rewrite the defining equation of the hyperplane as $x_d = -\frac{1}{a_d}(a_1x_1 + \ldots + a_{d-1}x_{d-1} - \alpha) = b_1x_1 + \ldots + b_{d-1}x_{d-1} - \beta$, where $b_i = -a_i/a_d$, $1 \leq i \leq d-1$, and $\beta = -\alpha/a_d$. We define a *duality transform* that sends points to nonverical hyperplanes, and vice versa:

For a point $p = (p_1, \ldots, p_d)$, we define

$$p^* := \{x \in \mathbb{R}^d : x_d = p_1x_1 + \ldots + p_{d-1}x_{d-1} - p_d\}.$$

Conversely, for a nonvertical hyperplane $h = \{x \in \mathbb{R}^d : x_d = b_1x_1 + \ldots + b_{d-1}x_{d-1} - \beta\}$, we set

$$h^* := (b_1, \ldots, b_{d-1}, \beta).$$

(a) Check that this is indeed a duality, in the following sense: For all points p and nonvertical hyperplanes h, $(p^*)^* = p$ and $(h^*)^* = h$.

(b) For a nonvertical hyperplane $h = \{x \in \mathbb{R}^d : x_d = b_1x_1 + \ldots + b_{d-1}x_{d-1} - \beta\}$, we adopt the following conventions for the corresponding halfspaces: $h^+ = \{x \in \mathbb{R}^d : x_d \geq b_1x_1 + \ldots + b_{d-1}x_{d-1} - \beta\}$ and $h^- = \{x \in \mathbb{R}^d : x_d \leq b_1x_1 + \ldots + b_{d-1}x_{d-1} - \beta\}$. Show that for all points p and all nonvertical hyperplanes h,

$$p \in \left\{ \begin{array}{c} h^+ \\ h^- \\ h \end{array} \right\} \quad \Leftrightarrow \quad h^* \in \left\{ \begin{array}{c} (p^*)^+ \\ (p^*)^- \\ p^* \end{array} \right\}.$$

Exercise 3

Show that without the additional compactness assumption, the infinite version of Helly's Theorem is generally not true. That is, give an example, for some dimension d of your choice, of an infinite family $\mathcal{C}$ of (noncompact) convex sets in $\mathbb{R}^d$ such that

(i) any $d+1$ of the sets in $\mathcal{C}$ have a nonempty intersection,

(ii) but $\bigcap_{C \in \mathcal{C}} C = \emptyset$.

Exercise 4

Let C, D be nonempty compact convex sets in $\mathbb{R}^d$.

(a) Show that there exist points $p \in C$ and $q \in D$ such that, for all $x \in C$ and all $y \in D$, $\|p-q\| \leq \|x-y\|$. (*Hint:* You may use the fact that $C \times D$ is also compact; which theorems about continuous functions on compact sets do you remember from analysis?)

(b) Let h be the hyperplane with normal vector $p - q$ and passing through the point $m := (p + q)/2$ (the midpoint of the segment pq; what is the equation of this hyperplane?). Show that h *separates* C and D, i.e., that $C \subseteq h^+$ and $D \subseteq h^-$. (In order to avoid confusion, we could let the hyperplane pass through any point on the segment pq and the statement would still be true.)