Eidgenössische Technische Hochschule Zürich
Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich

Institut für Theoretische Informatik
Lecturers: Bernd Gärtner, Uli Wagner, Emo Welzl
Assistant: Eva Schuberth

May 30, 2006

Approximate Methods in Geometry *SoSe 2006*

Exercise Set 9

Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/ApproxGeom06/`

Due date: June 6, 2006

Exercise 1

For real numbers a,b, and c, we define the region

$$p(a,b,c) := \{(x,y) \in \mathbb{R}^2 | y^2 \geq a(x-b)^2 + c\}$$

Now consider the range space $(\mathbb{R}^2, \mathcal{P})$ with $\mathcal{P} := \{p(a,b,c) | (a,b,c) \in \mathbb{R}^3\}$.

(1) Give three points that are shattered by $\mathcal{P}$ (give the coordinates of the points, and the specifying triples for the ranges used for the shattering).
(2) Show that the VC-dimension of the range space is 3. (Hint: Employ linearization, i.e. use a map similar to the lifting map ...)

Exercise 2

Let $(X, \mathcal{R})$ be a range space of finite VC-dimension d. Every pair of ranges q and r partitions X into four sets: $X \setminus (q \cup r)$, $q \cap r$, $q \setminus r$, and $r \setminus q$. Now let $\mathcal{R}'$ be the set of ranges obtained by taking for every pair of ranges the four sets as above as ranges into $\mathcal{R}'$. Find a concrete upper bound on the VC-dimension of $(X, \mathcal{R}')$ in terms of d. We leave it to your energy and ambitions, whether you want to get this bound as good as possible.

Exercise 3

Let $n \geq 5$. Suppose we are given a set P of n points in general position in the plane, i.e. no three points on a common line. We want to show that there is a positive constant c (independent from n) such that a random subset of 4 points in P is in convex position (i.e. the four points are vertices of a convex quadrilateral) with probability at least c. To be precise, we let F be a set uniformly at random in $\binom{P}{4}$. We want to show that

prob(F is in convex position) $\geq c$.

In order to do so proceed in two steps.
(1) Show that for every set P of five points the statement is true — for which constant c? (Hint: Use here that the complete graph on 5 vertices is not planar. Therefore, the complete graph with straight line edges on a set of five points has at least one crossing. What does the crossing tell us?)
(2) Now observe that for a general set P of more than 5 points, we can choose F by first choosing a set Q of 5 points uniformly at random in $\binom{P}{5}$ and then taking F as a random 4-element subset of Q (basically removing a point uniformly at random in Q). But after choosing Q we can use (1).

Remark: This exercise seems to have nothing to do with ε-nets — and so it is. However the idea of the argument follows the lines of the ε-net proof: First choose a small subset (or sequence), and then take a subset of the subset (or permute and consider some initial segment of the sequence). This is a strong paradigm for probabilistic arguments.