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Approximate Methods in Geometry *Spring 2007*

Exercise Set 1

Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/ApproxGeom07/`

Due date: March 27, 2007

Exercise 1

As a warm-up, show that if $\mathcal{C}$ is an arbitrary collection of convex sets in $\mathbb{R}^d$, then their intersection $\bigcap_{C\in\mathcal{C}} C$ is also convex. Next, prove that for any set $X \subseteq \mathbb{R}^d$,

$$\mathrm{conv}(X) = \{\sum_{x\in S} \lambda_x x \mid S \subset X \text{ finite}, \sum_{x\in S} \lambda_x = 1, \text{ and } \lambda_x \geq 0 \text{ for all } x \in S\}.$$

Exercise 2

A hyperplane $h = \{x \in \mathbb{R}^d : \langle a, x\rangle = \alpha\}$ is called *nonvertical* if $a_d \neq 0$, where $a = (a_1, \ldots, a_d)$ is the normal vector. Observe that we can rewrite the defining equation of the hyperplane as $x_d = -\frac{1}{a_d}(a_1x_1 + \ldots + a_{d-1}x_{d-1} - \alpha) = b_1x_1 + \ldots + b_{d-1}x_{d-1} - \beta$, where $b_i = -a_i/a_d$, $1 \leq i \leq d-1$, and $\beta = -\alpha/a_d$. We define a *duality transform* that sends points to nonverical hyperplanes, and vice versa:

For a point $p = (p_1, \ldots, p_d)$, we define

$$p^* := \{x \in \mathbb{R}^d : x_d = p_1x_1 + \ldots + p_{d-1}x_{d-1} - p_d\}.$$

Conversely, for a nonvertical hyperplane $h = \{x \in \mathbb{R}^d : x_d = b_1x_1 + \ldots + b_{d-1}x_{d-1} - \beta\}$, we set

$$h^* := (b_1, \ldots, b_{d-1}, \beta).$$

(a) Check that this is indeed a duality, in the following sense: For all points p and nonvertical hyperplanes h, $(p^*)^* = p$ and $(h^*)^* = h$.

(b) For a nonvertical hyperplane $h = \{x \in \mathbb{R}^d : x_d = b_1x_1 + \ldots + b_{d-1}x_{d-1} - \beta\}$, we adopt the following conventions for the corresponding halfspaces: $h^+ = \{x \in \mathbb{R}^d : x_d \geq b_1x_1 + \ldots + b_{d-1}x_{d-1} - \beta\}$ and $h^- = \{x \in \mathbb{R}^d : x_d \leq b_1x_1 + \ldots + b_{d-1}x_{d-1} - \beta\}$. Show that for all points p and all nonvertical hyperplanes h,

$$p \in \left\{\begin{array}{c} h^+ \\ h^- \\ h \end{array}\right\} \quad \Leftrightarrow \quad h^* \in \left\{\begin{array}{c} (p^*)^+ \\ (p^*)^- \\ p^* \end{array}\right\}.$$

Exercise 3

Show that without the additional compactness assumption, the infinite version of Helly's Theorem is generally not true. That is, give an example, for some dimension d of your choice, of an infinite family $\mathcal{C}$ of (noncompact) convex sets in $\mathbb{R}^d$ such that

(i) any $d+1$ of the sets in $\mathcal{C}$ have a nonempty intersection,

(ii) but $\bigcap_{C\in\mathcal{C}} C = \emptyset$.

Exercise 4

Let C, D be nonempty compact convex sets in $\mathbb{R}^d$, $C \cap D = \emptyset$.

(a) Show that there exist points $p \in C$ and $q \in D$ such that, for all $x \in C$ and all $y \in D$, $\|p-q\| \leq \|x-y\|$. (*Hint:* You may use the fact that $C \times D$ is also compact; which theorems about continuous functions on compact sets do you remember from analysis?)

(b) Let h be the hyperplane with normal vector $p - q$ and passing through the point $m := (p+q)/2$ (the midpoint of the segment pq; what is the equation of this hyperplane?). Show that h *separates* C and D, i.e., that $C \subseteq h^+$ and $D \subseteq h^-$. (Just to avoid confusion: There is nothing special about the midpoint, we could let the hyperplane pass through any other point in the interior of the segment pq and the statement would still be true.)