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Approximate Methods in Geometry *Spring 2007*

Exercise Set 2

Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/ApproxGeom07/`

Due date: April 3, 2007

Exercise 1

Let S be a finite set of vertical *line segments*[1] in $\mathbb{R}^2$, see Figure 1. Prove the following statement: If for every three of the line segments, there is a line that intersects all three segments, then there is a line that intersects *all* segments. (If you get stuck, try the hint on the back page.)

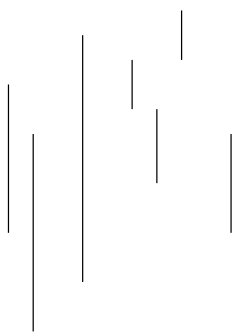

Figure 1: A set of vertical line segments in $\mathbb{R}^2$

Recall the following notation: For $1 \leq p \leq \infty$, ℓ_p^d denotes the real vector space $\mathbb{R}^d$ with the norm $\|x\|_p$. For $1 \leq p < \infty$, this norm is defined as $\|x\|_p = \sqrt[p]{\sum_{i=1}^d |x_i|^p}$; in the limit case $p = \infty$, $\|x\|_\infty = \max_{1 \leq i \leq d} |x_i|$.

Exercise 2

Show that for every $d \geq 1$, there is an isometry $f : \ell_1^d \to \ell_\infty^{2^d}$. (An isometry is a distance preserving or distortion 1 map, i.e., in our case we require that $\|f(x) - f(y)\|_\infty = \|x - y\|_1$ hold for all $x, y \in \mathbb{R}^d$.)

Exercise 3

Suppose you wish to design an algorithm that solves the following problem: Given a finite set $X \subseteq \ell_1^d$, $|X| = n$, compute the diameter $\operatorname{diam}(X) := \max_{x,y \in X} \|x - y\|_1$. What is the runtime of the "naive" algorithm that just computes pairwise distances? Show, using Exercise 2, that there is an algorithm that computes the diameter of a set of n points in ℓ_1^d in time $O(d2^d n)$. (If d is fixed and n is large, this improves upon the naive algorithm.)

[1] a line segment is the convex hull of a set of two points

Recall that a real-valued random variable X has *subgaussian tail* up to some number $\lambda_0 > 0$ if there is a constant $a > 0$ such that

$$\Pr[X > \lambda] \leq e^{-a\lambda^2}$$

for $0 < \lambda \leq \lambda_0$.

Exercise 4

Let X be a real-valued random variable with $\mathbf{E}[X] = 0$. Show that if

$$\mathbf{E}[e^{uX}] \leq e^{Cu^2}$$

holds for some constant C and all $0 < u \leq u_0$ for some $u_0 > 0$, then X has a subgaussian tail up to $2Cu_0$.

Hint for Exercise 1. Use the duality transform (non-vertical case) and Helly's Theorem. For this, you need to understand the following: (i) what is the set of lines dual to the set of points on a (vertical) segment? (ii) if a line intersects the segment, what can we say about the point dual to this line?