Exercise 1

Let $S$ be a finite set of vertical line segments in $\mathbb{R}^2$, see Figure 1. Prove the following statement: If for every three of the line segments, there is a line that intersects all three segments, then there is a line that intersects all segments. (If you get stuck, try the hint on the back page.)

Recall the following notation: For $1 \leq p \leq \infty$, $\ell^d_p$ denotes the real vector space $\mathbb{R}^d$ with the norm $\|x\|_p = \sqrt[p]{\sum_{i=1}^{d} |x_i|^p}$; in the limit case $p = \infty$, $\|x\|_\infty = \max_{1 \leq i \leq d} |x_i|$.

Exercise 2

Show that for every $d \geq 1$, there is an isometry $f : \ell^d_1 \to \ell^d_\infty$. (An isometry is a distance preserving or distortion 1 map, i.e., in our case we require that $\|f(x) - f(y)\|_\infty = \|x - y\|_1$ hold for all $x, y \in \mathbb{R}^d$.)

Exercise 3

Suppose you wish to design an algorithm that solves the following problem: Given a finite set $X \subseteq \ell^d_1$, $|X| = n$, compute the diameter $\text{diam}(X) := \max_{x, y \in X} \|x - y\|_1$. What is the runtime of the “naive” algorithm that just computes pairwise distances? Show, using Exercise 2, that there is an algorithm that computes the diameter of a set of $n$ points in $\ell^d_1$ in time $O(d^2 n)$. (If $d$ is fixed and $n$ is large, this improves upon the naive algorithm.)

\[1\] a line segment is the convex hull of a set of two points
Recall that a real-valued random variable $X$ has \textit{subgaussian tail} up to some number $\lambda_0 > 0$ if there is a constant $a > 0$ such that
\[ \Pr[|X| > \lambda] \leq e^{-a\lambda^2} \]
for $0 < \lambda \leq \lambda_0$.

\textbf{Exercise 4}

Let $X$ be a real-valued random variable with $\mathbb{E}[X] = 0$. Show that if
\[ \mathbb{E}[e^{uX}] \leq C e^{Cu^2} \]
holds for some constant $C$ and all $0 < u \leq u_0$ for some $u_0 > 0$, then $X$ has a subgaussian tail up to $2Cu_0$.

\textbf{Hint for Exercise 1.} Use the duality transform (non-vertical case) and Helly’s Theorem. For this, you need to understand the following: (i) what is the set of lines dual to the set of points on a (vertical) segment? (ii) if a line intersects the segment, what can we say about the point dual to this line?