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Approximate Methods in Geometry *Spring 2007*

Exercise Set 5

Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/ApproxGeom07/`

Due date: April 24, 2007

Exercise 1

Let $G = (V, E)$ be a graph on $|V| = n$ vertices. Consider the *adjacency matrix* $A_G = [a_{uv}] \in \mathbb{R}^{V \times V}$ defined by $a_{uv} = 1$ if $\{u, v\} \in E$ and $a_{uv} = 0$ otherwise. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of A_G (note that A_G is symmetric, so its eigenvalues are real, and we can pick corresponding eigenvectors $b_i \in \mathbb{R}^V$ that form an orthonormal basis).

(a) Show that $\lambda_1 \leq \Delta(G)$, where $\Delta(G)$ is the maximum degree of any vertex in G.

(b) Show that if G is r-regular (every vertex has degree exactly r), then $\lambda_1 = r$ with corresponding eigenvector $\mathbf{1} = (1, \ldots, 1)$. Conclude from Part (a) that the Laplacian L_G of G is positive semidefinite with eigenvalues $\mu_i = r - \lambda_i$, $1 \leq i \leq r$.

Bonus Question: Show that in Part (b), if G is connected, then $\mu_2 > 0$.

Exercise 2

Let $G = (V, E)$ be an r-regular graph on n vertices and let $0 = \mu_1 \leq \mu_2 \ldots \leq \mu_n$ be the eigenvalues of the Laplacian of G. Show that for disjoint $A, B \subseteq V$, with $V = A \cup B$, the number of $e(A, B)$ of edges between A and B is at least

$$e(A, B) \geq \mu_2 \cdot \frac{|A||B|}{n}.$$

Deduce that the edge expansion of G satisfies $\Phi(G) \geq \mu_2/2$.

Exercise 3

Let (V, ρ) be a finite metric space. Show that there is an embedding $f : (V, \rho) \to (\mathbb{R}^d, \|\cdot\|_2)$ of distortion at most D for some target dimension d iff there is a positive definite matrix $Q = [q_{uv}] \in \mathbb{R}^{V \times V}$ that satisfies

$$\rho(u, v) \leq q_{uu} + q_{vv} - 2q_{uv} \leq D^2 \rho(u, v)$$

for all $u, v \in V$. Conclude that the minimum distrotion necessary for embedding (V, ρ) into any Euclidean space can be determined in polynomial time by solving a semidefinite program.