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Approximate Methods in Geometry *Spring 2007*

Solutions to Exercise Set 4

Exercise 3

(a) Let $B = [b_{ij}]$ be a real $(n \times n)$-matrix such that $b_{ii} = 1$ for all i and $|b_{ij}| \le \varepsilon$ for $i \ne j$, where $1/\sqrt{n} \le \varepsilon \le 1/2$. Show that

$$\operatorname{rank}(B) \ge \Omega\left(\frac{\log n}{\varepsilon^2 \log(1/\varepsilon)}\right).$$

(b) Consider the set $X = \{0, e_1, \ldots, e_n\} \subset \mathbf{R}^n$ (where the e_i's are the vectors of the standard orthonormal basis). Suppose that this set of points (with their Euclidean distances) can be mapped with distortion at most $(1+\varepsilon)$ into ℓ_2^k (i.e., into $\mathbb{R}^k$ with Euclidean distances). Show that then there exist $v_1, \ldots, v_n \in \mathbb{R}^k$ that are "almost orthogonal" unit vectors, i.e., $\|v_i\|_2 = 1$ for all i and $|\langle v_i, v_j\rangle| \le 100\varepsilon$ (the constant 100 could be improved).

(c) Assuming that there is a low-distortion map as in Part (b) and $\frac{1}{100\sqrt{n}} \le \varepsilon \le 1/200$, show that

$$k \ge \Omega\left(\frac{\log n}{\varepsilon^2 \log(1/\varepsilon)}\right).$$

Proof of Part (a). Let $k := \frac{\log(n)}{2\log(1/\varepsilon)}$ (for simplicity, we ignore rounding up or down and assume that k is an integer). We have $\varepsilon^k = 1/\sqrt{n}$. We remark for later use that $k \ge 1$ and so $1/k \le 1$. Let $C = [c_{ij}]$ be the matrix defined by $c_{ij} := b_{ij}^k$. Then, C satisfies the assumptions of Exercise 1(b), hence $\operatorname{rank}(C) \ge n/4$. On the other hand, if $r := \operatorname{rank}(B)$, then by Exercise 2, $n/4 \le \operatorname{rank}(C) \le \left(\frac{e(k+r)}{k}\right)^k$. Solving for r, we obtain $r \ge k\left(\frac{1}{e}(n/4)^{1/k} - 1\right)$. Moreover, $(n/4)^{1/k} = n^{\frac{2\log(1/\varepsilon)}{\log n}} 4^{-1/k} \ge \varepsilon^{-2} 4^{-1/k} \ge \varepsilon^{-2}/4$ since $-1/k \ge -1$. Therefore, $r \ge \Omega(\frac{\log n}{\varepsilon^2 \log(1/\varepsilon)})$. □

Proof of Part (b). Let $w_0, w_1, \ldots, w_n$ be the images of $0, e_1, \ldots, e_n$ under the low-distortion map. By a suitable translation, if necessary, we may assume that $w_0 = \mathbf{0}$. Thus, we have $\mathbf{1} \le |w_i\|_2 \le 1+\varepsilon$ for $1 \le i \le n$ and $\sqrt{2} \le \|w_i - w_j\|_2 \le (1+\varepsilon)\sqrt{2}$ for $i \ne j$. We set $v_i := w_i/\|w_i\|_2$ for $1 \le i \le n$. Then the v_i's are unit vectors. Note that we can write $v_i = w_i - r_i$, where $r_i = (\|w_i\|_2 - 1)v_i$ is a vector of length at most ε. Therefore, by the triangle inequality, we have $\sqrt{2} - 2\varepsilon \le \|v_i - v_j\|_2 \le (1+\varepsilon)\sqrt{2} + 2\varepsilon$ for $i \ne j$. On the other hand, we have $\|v_i - v_j\|_2^2 = \|v_i\|_2^2 + \|v_j\|_2^2 - 2\langle v_i, v_j\rangle$. It follows that $-(4\sqrt{2}(2+\varepsilon) + 4\varepsilon)\varepsilon \le 2\langle v_i, v_j\rangle \le 4\sqrt{2}\varepsilon - 4\varepsilon^2$, so $|\langle v_i, v_j\rangle| < 100\varepsilon$, *say*. □

Proof of Part (c). We use Parts (a) and (b): Define a matrix B by $b_{ij} := \langle v_i, v_j\rangle$, with unit vectors $v_i \in \mathbb{R}^k$ as in Part (b). Observe that $\operatorname{rank}(B) \le k$. Moreover, the matrix B satisfies the assumptions of Part (a) (with $\varepsilon' = 100\varepsilon$), so $k \ge\ge \Omega(\frac{\log n}{\varepsilon^2 \log(1/\varepsilon)})$ as desired. □