Approximate Methods in Geometry Spring 2007

Exercise 3

(a) Let $B = [b_{ij}]$ be a real $(n \times n)$-matrix such that $b_{ii} = 1$ for all $i$ and $|b_{ij}| \leq \varepsilon$ for $i \neq j$, where $1/\sqrt{n} \leq \varepsilon \leq 1/2$. Show that

$$\text{rank}(B) \geq \Omega \left( \frac{\log n}{\varepsilon^2 \log(1/\varepsilon)} \right).$$

(b) Consider the set $X = \{0, e_1, \ldots, e_n\} \subset \mathbb{R}^n$ (where the $e_i$’s are the vectors of the standard orthonormal basis). Suppose that this set of points (with their Euclidean distances) can be mapped with distortion at most $(1 + \varepsilon)$ into $\ell_2^n$ (i.e., into $\mathbb{R}^k$ with Euclidean distances). Show that then there exist $v_1, \ldots, v_n \in \mathbb{R}^k$ that are “almost orthogonal” unit vectors, i.e., $\|v_i\| = 1$ for all $i$ and $|\langle v_i, v_j \rangle| \leq 100\varepsilon$ (the constant 100 could be improved).

(c) Assuming that there is a low-distortion map as in Part (b) and $\frac{\log n}{\log \sqrt{n}} \leq \varepsilon \leq 1/200$, show that

$$k \geq \Omega \left( \frac{\log n}{\varepsilon^2 \log(1/\varepsilon)} \right).$$

Proof of Part (a). Let $k := \frac{\log(n)}{2 \log(1/\varepsilon)}$ (for simplicity, we ignore rounding up or down and assume that $k$ is an integer). We have $\varepsilon^k = 1/\sqrt{n}$. We remark for later use that $k \geq 1$ and so $1/k \leq 1$. Let $C = [c_{ij}]$ be the matrix defined by $c_{ij} := b_{ij}^k$. Then, $C$ satisfies the assumptions of Exercise 1(b), hence $\text{rank}(C) \geq n/4$.

On the other hand, if $r := \text{rank}(B)$, then by Exercise 2, $n/4 \leq \text{rank}(C) \leq \left( \frac{\varepsilon(k+1)}{k} \right)^k$. Solving for $r$, we obtain $r \geq k \left( \frac{1}{e} (n/4)^{1/k} - 1 \right)$. Moreover, $(n/4)^{1/k} = n^{\frac{1}{2\log(1/\varepsilon)}} 4^{-1/k} \geq \varepsilon^{-2}4^{-1/k} \geq \varepsilon^{-2}/4$ since $-1/k \geq -1$. Therefore, $r \geq \Omega \left( \frac{\log n}{e^2 \log(1/\varepsilon)} \right)$.

Proof of Part (b). Let $w_0, w_1, \ldots, w_n$ be the images of $0, e_1, \ldots, e_n$ under the low-distortion map. By a suitable translation, if necessary, we may assume that $w_0 = 0$. Thus, we have $1 \leq \|w_i\| \leq 1 + \varepsilon$ for $1 \leq i \leq n$ and $\sqrt{2} \leq \|w_i - w_j\| \leq (1 + \varepsilon)\sqrt{2}$ for $i \neq j$. We set $v_i := w_i/\|w_i\|$ for $1 \leq i \leq n$. Then the $v_i$’s are unit vectors. Note that we can write $v_i = w_i - r_i$, where $r_i = (\|w_i\| - 1)v_i$ is a vector of length at most $\varepsilon$. Therefore, by the triangle inequality, we have $\sqrt{2} - 2\varepsilon \leq \|v_i - v_j\| \leq (1 + \varepsilon)\sqrt{2} + 2\varepsilon$ for $i \neq j$. On the other hand, we have $\|v_i - v_j\| = \|v_i\|^2 + \|v_j\|^2 - 2\|v_i, v_j\|$. It follows that $-4\sqrt{2}(2 + \varepsilon + 4\varepsilon)\varepsilon \leq 2\|v_i, v_j\| \leq 4\sqrt{2}(1 + \varepsilon)$, so $|\langle v_i, v_j \rangle| \leq 100\varepsilon$, say.

Proof of Part (c). We use Parts (a) and (b): Define a matrix $B$ by $b_{ij} := \langle v_i, v_j \rangle$, with unit vectors $v_i \in \mathbb{R}^k$ as in Part (b). Observe that $\text{rank}(B) \leq k$. Moreover, the matrix $B$ satisfies the assumptions of Part (a) (with $\varepsilon' = 100\varepsilon$), so $k \geq \Omega \left( \frac{\log n}{e^2 \log(1/\varepsilon)} \right)$ as desired.