

**Approximate Methods in Geometry**      **Spring 2008****Exercise Set 2**

Course Webpage: <http://www.ti.inf.ethz.ch/ew/courses/ApproxGeom08/>

Due date: March 4, 2008

**Exercise 1**

Let  $S$  be a finite set of vertical *line segments*<sup>1</sup> in  $\mathbb{R}^2$ , see Figure 1. Prove the following statement: If for every three of the line segments, there is a line that intersects all three segments, then there is a line that intersects *all* segments. (If you get stuck, try the hint on the back page.)

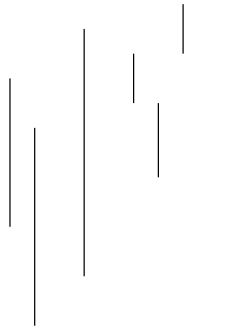


Figure 1: A set of vertical line segments in  $\mathbb{R}^2$

**Exercise 2**

Let  $A = [a_{ij}]$  be a symmetric  $(n \times n)$ -matrix.  $A$  is called *positive semidefinite*, written  $A \succcurlyeq 0$ , if  $x^T A x \geq 0$  holds for all  $x \in \mathbb{R}^n$  (where  $\cdot^T$  denotes the transpose).

(a) Show that the set  $\text{SDF}_n$  of positive semidefinite symmetric  $(n \times n)$ -matrices is a convex cone, i.e., if  $A, B \in \text{SDF}_n$  and  $\alpha, \beta \in \mathbb{R}_{\geq 0}$ , then  $\alpha A + \beta B \in \text{SDF}_n$ .

(b) Show that the following statements are equivalent:

- (i)  $A \succcurlyeq 0$ .
- (ii) All eigenvalues of  $A$  are nonnegative.
- (iii) For some dimension  $d$ , there exist vectors  $u_1, \dots, u_n \in \mathbb{R}^d$  such that  $A$  is the *Gram matrix* of the  $u_i$ 's, in the sense that  $a_{ij} = \langle u_i, u_j \rangle$  for  $1 \leq i, j \leq n$ . In other words,  $A = U^T U$  for some  $U \in \mathbb{R}^{d \times n}$ . (A bonus question: What is the minimal  $d$  (as a well-known function of  $A$ ) such that  $A$  can be written as such a Gram matrix?)
- (iv)  $A$  can be written as a nonnegative linear combination of matrices of the form  $vv^T$ ,  $v \in \mathbb{R}^n$ .

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<sup>1</sup>a line segment is the convex hull of a set of two points

### Exercise 3

The topic of this exercise is another application of semidefinite programming, this time to MAX-2-SAT. Recall that a 2-CNF formula in the variables  $x_1, \dots, x_n$  is a formula in *conjunctive normal form*  $\varphi = C_1 \wedge \dots \wedge C_m$ , where each *clause*  $C_k = (\ell_i \vee \ell_j)$  is the disjunction of two literals  $\ell_i \in \{x_1, \dots, x_n\} \cup \{\neg x_1, \dots, \neg x_n\}$ , e.g. example,

$$\varphi = (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_2 \vee \neg x_3).$$

MAX-2-SAT is the following problem: given a 2-CNF formula  $\varphi$ , find an assignment  $x_i \mapsto \{\mathbf{true}, \mathbf{false}\}$  of truth values to the variables that satisfies as many clauses as possible.

(a) Describe a randomized approximation algorithm, which in expectation achieves an approximation ratio of  $1/2$ .

(b) Reformulate MAX-2-SAT as a quadratic programming problem in variables  $y_i \in \{+1, -1\}$ . *Hint:* In order to circumvent certain technical difficulties, introduce an additional variable  $y_0$  and interpret  $x_i = \mathbf{true}$  as  $y_i = y_0$ , and  $x_i = \mathbf{false}$  as  $y_i \neq y_0$ , and proceed as for MAXCUT. (In order to appreciate the hint, you might first want to try it without  $y_0$ .)

(c) Define a SDP relaxation of the above quadratic program and show how to use it to obtain an  $(0.878 - \varepsilon)$ -approximation- algorithm for MAX-2-SAT.

### Exercise 4 (Bonus)

Define

$$\alpha := \inf_{-1 \leq s < 1} \frac{2 \arccos(s)}{\pi(1-s)} = \inf_{0 < t \leq \pi} \frac{2t}{\pi(1 - \cos(t))}.$$

Show that  $\alpha > 0.87856$ . Since this will in any case involve a computer calculation, you are allowed to use any mathematical software.

(To avoid ambiguities, we mean the function  $\arccos : [-1, 1] \rightarrow [0, \pi]$  that is the inverse of the cosine function  $\cos : [0, \pi] \rightarrow [-1, 1]$ .)

**Hint for Exercise 1.** Use the duality transform (non-vertical case) and Helly's Theorem. For this, you need to understand the following: (i) what is the set of lines dual to the set of points on a (vertical) segment? (ii) if a line intersects the segment, what can we say about the point dual to this line?