

SOLUTION TO EXERCISE 2.3 B,C

We will proceed in a similar way as with the MAX-CUT in the lecture.

**b).** We will encode the variables of the formula  $(x_i)_{i=1}^n$  through  $\pm 1$  variables  $y_i$  of an optimization problem  $(y_j)_{j=0}^n$ ,  $y_j \in \{-1, +1\}$  in such a way that  $x_i = \text{true}$  when  $y_i = y_0$  and false otherwise ( $y_i = -y_0$ ). Arithmetization  $a(l)$  of a literal is then:

$$a(l) = \begin{cases} l = x_i & \frac{1+y_0y_i}{2} \\ l = \neg x_i & \frac{1-y_0y_i}{2} \end{cases}. \text{ One can easily check that } a(x_i) = \begin{cases} 0 & y_0 \neq y_i \\ 1 & y_0 = y_i \end{cases}$$

and  $a(\neg x_i) = \begin{cases} 0 & y_0 = y_i \\ 1 & y_0 \neq y_i \end{cases}$ , which corresponds to the meaning of the variables explained above. We can extend this to the clauses. Observe, that  $a(\neg l) = 1 - a(l)$ .

If  $C = (l_1 \vee l_2)$  is a clause, then define  $a(C) := 1 - a(\neg l_1) \cdot a(\neg l_2)$ . One can again easily check, that  $a(C) = 1$  if and only if at least one of the literals is satisfied, i.e.  $a(l_1) = 1$  or  $a(l_2) = 1$ . More precisely written,

$$a(C) = 1 - \frac{1 + s_1 y_0 y_i}{2} \cdot \frac{1 + s_2 y_0 y_j}{2}$$

where  $s_1, s_2 \in \{-1, 1\}$  are constants determining the signs of the literals  $l_1, l_2$  in  $C$ . This can be rewritten as

$$\begin{aligned} a(C) &= \frac{3 + s_1 y_0 y_i + s_2 y_0 y_j + s_1 s_2 y_i y_j y_0^2}{4} \\ &= \frac{1 + s_1 y_0 y_i}{4} + \frac{1 + s_2 y_0 y_j}{4} + \frac{1 + s_1 s_2 y_i y_j}{4} \end{aligned}$$

since  $y_0^2 = 1$ , which is a quadratic function in  $y$ .

With this arithmetization, we can express the number of satisfied clauses  $s(\phi)$  by simply summing up  $a(C)$  over all the clauses. Maximizing this expression gives us a quadratic optimization problem of the form

$$\max \left\{ \sum_{i,j \in [n]} a_{i,j} \frac{1 + y_i y_j}{2} + \sum_{i,j \in [n]} b_{i,j} \frac{1 - y_i y_j}{2} \mid y_i \in \{-1, 1\} \right\}$$

for some positive constants  $a_{i,j}$  and  $b_{i,j}$  (from the summation of previous).

**c).** The SDP relaxation follows similarly as in the lecture. Instead of  $y_i \in \{-1, 1\}$  we take  $u_i \in \mathbb{R}^n$ ,  $u_i^T u_i = 1$  (i.e.  $u_i \in \mathbb{S}^n$ ). The relaxation is then

$$\max \left\{ \sum_{i,j \in [n]} a_{i,j} \frac{1 + u_i^T u_j}{2} + \sum_{i,j \in [n]} b_{i,j} \frac{1 - u_i^T u_j}{2} \mid \forall i : u_i^T u_i = 1 \right\}$$

Such vectors determine a positive semidefinite matrix  $M = (u_i^T u_j)_{i,j}$  and as we have learned in the earlier exercises each such matrix also determines the vectors  $u_i$ . This gives a semidefinite program

$$\max \left\{ \sum_{i,j \in [n]} a_{i,j} \frac{1 + m_{ij}}{2} + \sum_{i,j \in [n]} b_{i,j} \frac{1 - m_{ij}}{2} \mid \forall i : m_{ii} = 1, M \succeq 0 \right\}.$$

One can quickly check, that a feasible solution of the previous quadratic program gives a feasible solution with the same optimum for this SDP (so it is indeed a relaxation). To do so, given  $y_j$ , consider the vector  $u_j = (y_j, 0, \dots, 0)$ . Then  $u_i^T u_j = y_i y_j$  and such  $u$ 's define the desired feasible solution.

The approximation algorithm is the following:

- (1) Find an optimal solution  $M$  of the SDP
- (2) Calculate the Gram decomposition of  $M$  into the  $u_i$ 's
- (3) Do the randomized rounding procedure as described in the lecture. Choose  $v$  on a unit sphere uniformly at random and decode  $y_i := v^T u_i$ .

To prove that the expected objective value of the QP given by these  $y_i$ 's is at least  $\alpha$  times the optimum, it suffices to show that each summand is in expectation at least  $\alpha$  times its value in the SDP. The only difference from the lecture is that we have two types of summands, namely  $\frac{1-m_{ij}}{2}$  and  $\frac{1+m_{ij}}{2}$ . We need to show the following

**Lemma 1.** *Let  $u, u', v, x, x'$  as in the Lemma 2.4 of the lecture notes. Then*

- (1)  $\mathbb{E}[\frac{1-xx'}{2}] \geq \alpha \frac{1-u^T u'}{2}$
- (2)  $\mathbb{E}[\frac{1+xx'}{2}] \geq \alpha \frac{1+u^T u'}{2}$

The first part is exactly what Lemma 2.4 states.

$$\mathbb{E}[\frac{1+xx'}{2}] = Pr[xx' = 1] = \frac{\arccos(-u^T u')}{2} = \alpha \frac{1+u^T u'}{2}$$

The first equality is clear. The second one can be obtained as in the lemma with the difference, that we are interested in the complementary event, therefore the angle in question is the complement of the angle defined by  $u$  and  $u'$ , which is the angle defined by  $-u$  and  $u'$ . The third inequality is implied by the Lemma 2.4 itself. This proves the second part and thus, our algorithm gives an  $\alpha$  approximation.