Exercise 1
Show that for every $d \geq 1$, there is an isometry $f : \ell_1^d \to \ell_\infty^d$. (An isometry is a distance preserving or distortion 1 map, i.e., in our case we require that $||f(x) - f(y)||_\infty = ||x - y||_1$ hold for all $x, y \in \mathbb{R}^d$.)

Exercise 2
Suppose you wish to design an algorithm that solves the following problem: Given a finite set $X \subseteq \ell_1^d$, $|X| = n$, compute the diameter $\text{diam}(X) := \max_{x, y \in X} ||x - y||_1$. What is the runtime of the “naive” algorithm that just computes pairwise distances? Show, using Exercise 1, that there is an algorithm that computes the diameter of a set of $n$ points in $\ell_1^d$ in time $O(d^4n)$. (If $d$ is fixed and $n$ is large, this improves upon the naive algorithm.)

Recall that a real-valued random variable $X$ has subgaussian tail up to some number $\lambda_0 > 0$ if there is a constant $a > 0$ such that

$$\Pr[X > \lambda] \leq e^{-a\lambda}$$

for $0 < \lambda \leq \lambda_0$.

Exercise 3
Let $X$ be a real-valued random variable with $\mathsf{E}[X] = 0$. Show that if $\mathsf{E}[e^{uX}] \leq e^{Cu^2}$ holds for some constant $C$ and all $0 < u \leq u_0$ for some $u_0 > 0$, then $X$ has a subgaussian tail up to $2Cu_0$.

Optional Exercise 1
Let $X$ and $Y$ be random variables that take only finitely many values. Show that if $X$ and $Y$ are independent then $\mathsf{E}[XY] = \mathsf{E}[X] \cdot \mathsf{E}[Y]$.

Optional Exercise 2
Prove the following estimate for the exponential function:

$$\frac{e^x + e^{-x}}{2} \leq e^{x^2/2} \quad \text{for all } x \in \mathbb{R}$$