Exercise 1

Consider the lifting function $l : \mathbb{R}^d \to \mathbb{R}^{d+1}$ defined by $l(x_1, \ldots, x_d) = (x_1, \ldots, x_d, \sum_{i=1}^{d} x_i^2)$. Prove the following statement: Two point sets $P, Q \subseteq \mathbb{R}^d$ can be separated by a sphere (i.e. there exists $c \in \mathbb{R}^d$ and $r \in \mathbb{R}$ such that $\|p - c\| < r$ for $p \in P$ and $\|q - c\| > r$ for $q \in Q$ or vice versa) if and only if the point sets $l(P), l(Q) \subseteq \mathbb{R}^{d+1}$ can be separated by a hyperplane (meaning strict separation).

Exercise 2

Let $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be defined as $k(x, y) = (x^T y)^p$, where $p \in \mathbb{N}$ is some constant. Find a value $d'$ and a function $l : \mathbb{R}^{d'} \to \mathbb{R}^{d'}$ such that $k(x, y) = l(x)^T l(y)$.

Exercise Bonus

You are in a town that you do not know, you do not speak the local language and you need to find the train station. Your strategy then can be to deduce the location of the train station from the behavior of people with suitcases. You walk around the town in a random fashion, and whenever you see someone walking with a suitcase, you conclude that the train station must be nearby, but whenever you see someone with a suitcase waiting for a bus or hailing a taxi, you conclude that the station is far away. Given that you meet sufficiently many people of both kinds, can you eventually find the train station?

Note: as the question is not formulated completely clearly, do any nice interpretation of the problem in the light of things done on the lecture (which would give one a recipe to find the station, of course).