Models of Computation

1. Real RAM

    - store and compute with (exact) real numbers;

    - all arithmetic operations constant time;

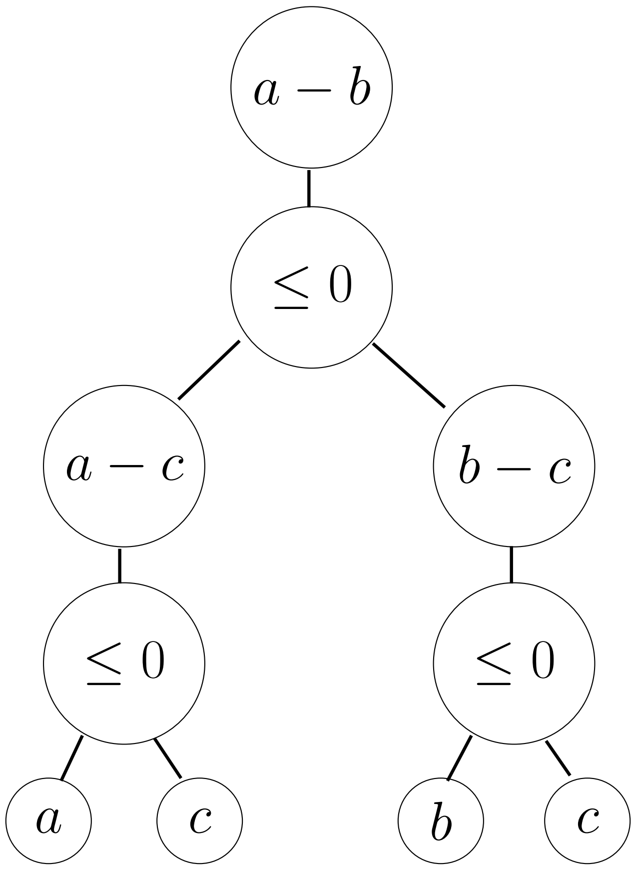
    - (integral) indirect addressing;

    - sometimes (square-)roots, logarithms, or other analytic functions;

    - sometimes floor and ceiling.

2. Algebraic Computation Trees

- computation is a binary tree;
- results are in the leaves;
- interior node with one child: operation $+, -, *, /, \sqrt{\cdot}, \ldots$ on two ancestors (or constants or input values);
- interior node with two children: branch of the form ≥ 0, ≤ 0, $= 0$.



Hulls

Let $P \subset \mathsf{R}^d$, $n \in \mathsf{N}$, $p_i \in P$ and $\lambda_i \in \mathsf{R}$ for $i \in \{1, \ldots, n\}$.

Linear Hull

$$\text{lin}(P) := \left\{ q \;\middle|\; q = \sum_{i=1}^{n} \lambda_i p_i \right\}$$

(Ex. $P = \{p\} \subset \mathsf{R}^2$: line through p and the origin.)

Affine Hull

$$\text{aff}(P) := \left\{ q \;\middle|\; q = \sum_{i=1}^{n} \lambda_i p_i \wedge \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

(Ex. $P = \{p, q\} \subset \mathsf{R}^2$: line through p and q.)

Convex Hull

$$\text{conv}(P) :=$$

$$\left\{ q \;\middle|\; q = \sum_{i=1}^{n} \lambda_i p_i \wedge \sum_{i=1}^{n} \lambda_i = 1 \wedge \forall\, i \;:\; \lambda_i \geq 0 \right\}$$

(Ex. $P = \{p, q\} \subset \mathsf{R}^2$: line segment $\overline{pq}$ through p and q.)

Convexity

Def. $P \subseteq \mathsf{R}^d$ is convex if and only if $\overline{pq} \subseteq P$ for any $p, q \in P$.

Obs. conv(P) is convex.

Let $p = \sum_{i=1}^{n} \lambda_i p_i$ and $q = \sum_{i=1}^{n} \mu_i p_i$, s.t. $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mu_i = 1$ and $\lambda_i, \mu_i \geq 0$ for all $i \in \{1, \ldots, n\}$.

For any $\alpha \in [0, 1]$

$$\begin{aligned} \alpha p + (1 - \alpha) q &= \alpha \sum_{i=1}^{n} \lambda_i p_i + (1 - \alpha) \sum_{i=1}^{n} \mu_i p_i \\ &= \sum_{i=1}^{n} (\alpha \lambda_i + (1 - \alpha) \mu_i) p_i \ , \end{aligned}$$

where $\sum_{i=1}^{n} (\alpha \lambda_i + (1 - \alpha) \mu_i) = \alpha + (1 - \alpha) = 1$, $\alpha \lambda_i \geq 0$, and $(1 - \alpha) \mu_i \geq 0$ for all $i \in \{1, \ldots, n\}$.

Characterization of Convexity

Thm. conv(P) is

1. the smallest convex subset of R^d containing P.

2. the intersection of all convex supersets of P.

3. the intersection of all closed halfspaces containing P.

In particular: P finite $\Rightarrow$ conv(P) is intersection of a finite number of halfspaces, that is, a convex polytope. [McMullen-Shephard 1971]

Thm. (Carathéodory) For $P \subset \mathsf{R}^d$ and $q \in$ conv(P) there exist $k \leq d+1$ points $p_1, \ldots, p_k \in P$ such that $q \in$ conv($p_1, \ldots, p_k$).

Constructing Convex Hulls in R^2

Convex Hull

Input: $P = \{p_1, \ldots, p_n\} \subset \mathsf{R}^2$, $n \in \mathsf{N}$.

Output: sequence $(q_1, \ldots, q_h)$, $1 \leq h \leq n$, of vertices of conv(P) (oriented ccw).

Extremal Points

Input: $P = \{p_1, \ldots, p_n\} \subset \mathsf{R}^2$, $n \in \mathsf{N}$.

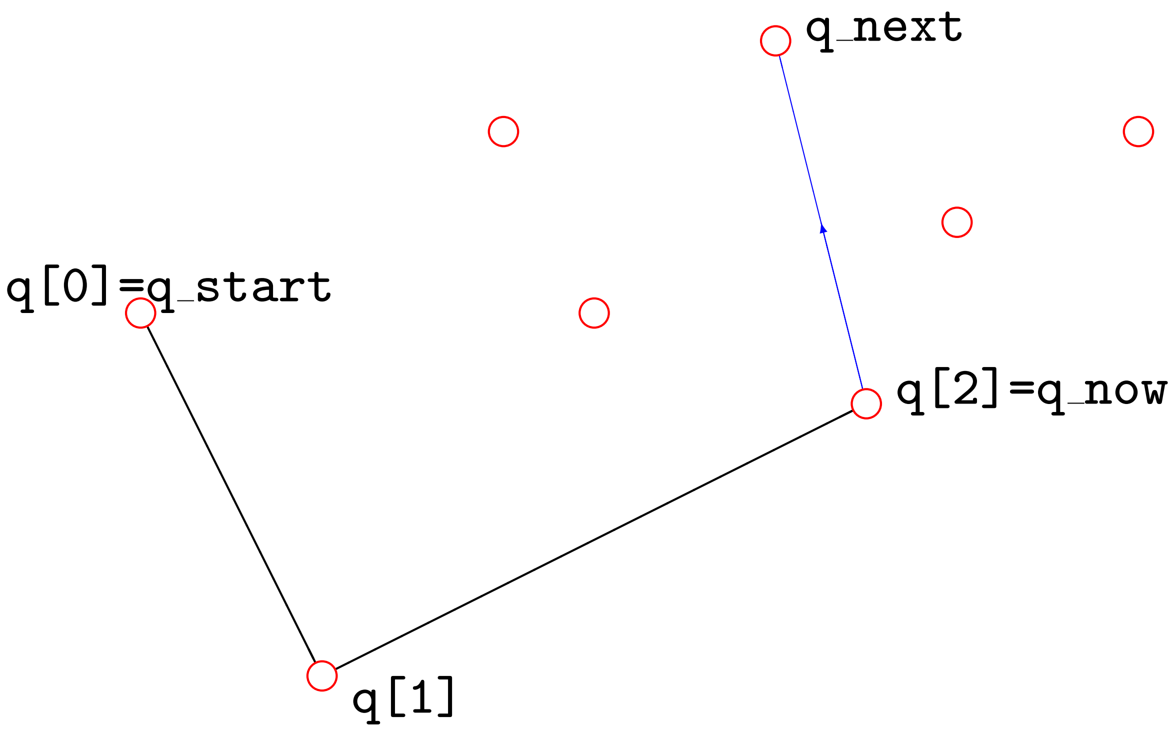
Output: set $\{q_1, \ldots, q_h\}$, $1 \leq h \leq n$, of vertices of conv(P).

What if three points in P are collinear?

Def. A point $p \in P = \{p_1, \ldots, p_n\} \subset \mathsf{R}^2$ is *vertex* of conv(P) $\iff$ there is a directed line g through p such that $P \setminus \{p\}$ is left of g.

Jarvis' Wrap

Find the point p_1 with smallest x-coordinate.
"Wrap" P starting from p_1 ccw
(Find the next point as the one from P that is furthest to the right.)
Until p_1 is reached again.



Jarvis' Wrap — Implemented

`p[0..N)` contains a sequence of points.
`p_start` point with smallest x-coordinate.
`q_next` some *other* point in `p[0..N)`.

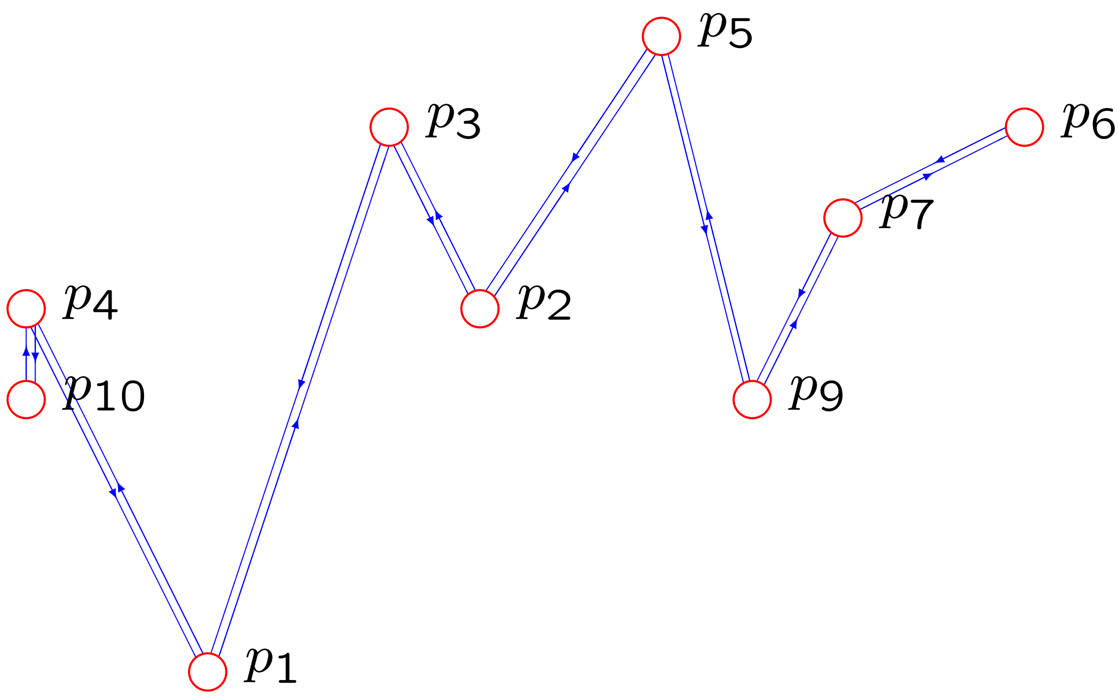
```
int h = 0;
Point_2 q_now = p_start;
do {
  q[h] = q_now;
  h = h + 1;

  for (int i = 0; i < N; i = i + 1)
    if (rightturn_2(q_now, q_next, p[i]))
      q_next = p[i];

  q_now = q_next;
  q_next = p_start;
} while (q_now != p_start);
```

`q[0]  q[1]  ...  q[h-1]`

describes a convex polygon bounding the convex hull of `p[0..N)`.

Graham Scan (SLR)

Sort points lexicographically and remove duplicates: $(p_1, \ldots, p_n)$.



$p_{10}\, p_4\, p_1\, p_3\, p_2\, p_5\, p_9\, p_7\, p_6\, p_7\, p_9\, p_5\, p_2\, p_3\, p_1\, p_4\, p_{10}$

Lower Convex Hull:
As long as there is a (consecutive) triple (p, q, r) s.t. q is left of or on the directed line $\overrightarrow{pr}$, remove q from the sequence.

Successive Local Repair — Implemented

`p[0..N)` lexicographically sorted sequence of pairwise distinct points, $N \geq 2$.

```
q[0] = p[0];
int k = 0;
// Lower convex hull (left to right):
for (int i = 1; i < N; i = 1 + 1) {
  while (k>0 && rightturn_2(q[k-1], q[k], p[i]))
    k = k - 1;
  k = k + 1;
  q[k] = p[i];
}

// Upper convex hull (right to left):
for (int i = N-2; i >= 0; i = i - 1) {
  while (rightturn_2(q[k-1], q[k], p[i]))
    k = k - 1;
  k = k + 1;
  q[k] = p[i];
}
```

`q[0..h)`, $h = k - 1$, describes a convex polygon with vertices from `p[0..N)` as they are encountered on the boundary of the convex hull.

Chan's Algorithm — Divide

Input: $P \subset \mathrm{R}^2$ with $|P| = n$ and a natural number $H \leq n$.

1. Divide P into $k = \lceil n/H \rceil$ sets $P_1, \ldots, P_k$ with $|P_i| \leq H$.

2. Construct conv($\mathrm{P_i}$) for all i, $1 \leq i \leq k$.

3. Construct H vertices of conv(P). (*conquer*)

Chan's Algorithm — Conquer

1. Find the lexicographically smallest point in conv(P$_i$) for all i, $1 \leq i \leq k$.

2. Starting from the lexicographically smallest point of P find the first H points of conv(P) oriented ccw (Jarvis' Wrap on the sequences conv(P$_i$)).

Determine in every step the points of tangency from the current point to of conv(P) to conv(P$_i$), $1 \leq i \leq k$, using linear search.