Post Office Problem

Given: $P \subset \mathbb{R}^2$, $|P| = n$.

Want: data structure to find the closest point from P for any given $q \in \mathbb{R}^2$.

Trivial:

- Query: $\mathcal{O}(n)$
- Preprocessing: $\mathcal{O}(1)$
- Space: $\mathcal{O}(1)$

Goal:

- Query: $\mathcal{O}(\log n)$
- Preprocessing: $\mathcal{O}(n \log n)$
- Space: $\mathcal{O}(n)$

Idea: Locus approach – Subdivide the domain into regions on which the answer is the same.

Voronoi Cells

Let $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$.

Definition 1 For $p_i \in P$ the Voronoi cell $V_P(i)$ of p_i with respect to P is

$$V_P(i) := \left\{ q \in \mathbb{R}^2 \;\middle|\; \|q - p_i\| \leq \|q - p\| \;\; \forall\, p \in P \right\}.$$

Observation 2

$$V_P(i) = \bigcap_{j \neq i} H(p_i, p_j) \text{ , where}$$

$$H(p_i, p_j) = \left\{ q \in \mathbb{R}^2 \;\middle|\; \|q - p_i\| \leq \|q - p_j\| \right\}.$$

Observation 3 $V_P(i) \neq \emptyset$ and $V_P(i)$ is convex.

Proof. $p_i \in V_P(i)$ and $V_P(i)$ is an intersection of a finite number of halfplanes. □

Observation 4 $V_P(i) \cap V_P(j)$ for $i \neq j$ is either empty or a line segment or a ray or a line.

Definition 5 (Voronoi Diagram)
VD(P) is the subdivision of the plane induced by the Voronoi polygons $V_P(i)$, $i = 1, \ldots, n$.

Denote by VV(P) the set of vertices, by VE(P) the edges and by VR(P) the regions (faces) of VD(P).

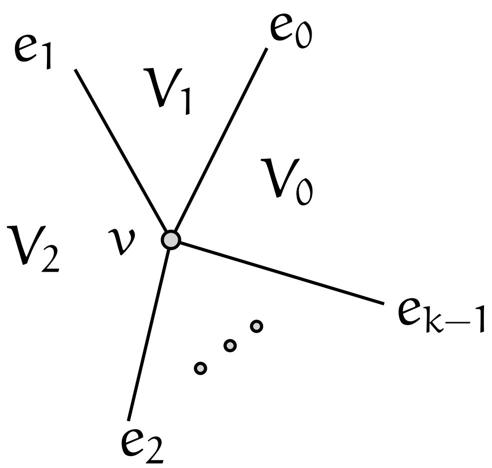
From now on we assume P is in general position, that is,

- no three points are collinear and
- no four points are cocircular.

Lemma 6 *For every vertex* $v \in \mathsf{VV}(P)$*:*

1. v *is intersection of exactly three edges from* $\mathsf{VE}(P)$*;*
2. v *is incident to exactly three regions from* $\mathsf{VR}(P)$*;*
3. v *is midpoint of a circle* $C(v)$ *through exactly three points from* P*;*
4. $\mathsf{Int}(C(v)) \cap P = \emptyset$.

Proof. Consider $v \in \mathsf{VV}(P)$.
Regions are convex $\Rightarrow$ $k \geq 3$ incident edges $e_0, \ldots, e_{k-1} \subseteq \mathsf{VE}(P)$.



$v \in e_i \Rightarrow |v - p_i| = |v - p_{(i+1) \bmod k}|$, that is, $p_1, p_2, \ldots, p_{k-1}$ are cocircular.

4) If $p_\ell \in \mathsf{Int}(C(v))$ then $v \in V_P(\ell)$ but not in $V_P(0), \ldots V_P(k-1)$, a contradiction. $\square$

Voronoi Diagram and Delaunay Triangulation

Lemma 7 The Voronoi edge between $V_P(i)$ and $V_P(j)$ is unbounded $\iff$ $\overline{p_i p_j}$ is an edge of conv(P).

Theorem 8 (Delaunay 1934) For a finite set $P \subset \mathbb{R}^2$ of $n \geq 3$ points in general position the straight-line dual of VD(P) is DT(P).

(Straight-line dual: Graph $G = (P, E)$;
$\overline{p_i p_j} \in E \iff |V(i) \cap V(j)| > 1 \wedge i \neq j$.)

Corollary 9 $|\mathrm{VE}(P)| \leq 3n - 6$, $|\mathrm{VV}(P)| \leq 2n - 5$, and VD(P) can be constructed in $\mathcal{O}(n \log n)$ time.

Voronoi Diagram and Delaunay Triangulation

Proof. (of Theorem 8)
Consider $v \in VV(P)$.
By Lemma 6 v is incident to 3 regions $V_P(i_v)$, $V_P(j_v)$ and $V_P(k_v)$. Let $T(v) := \triangle p_{i_v} p_{j_v} p_{k_v}$.

Claim:

$$\mathcal{T}(P) := \{T(v) \mid v \in VV(P)\}$$

is a triangulation of P.

(Delaunay property follows from Lemma 6d.)

1. If two triangles from $\mathcal{T}(P)$ intersect, they intersect in a common edge.
2. Every point from $CH(P)$ is contained in at least one triangle from $\mathcal{T}(P)$.

□

Lifting Map revisited

Definition 10 Let

$$\mathcal{U} = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; z = x^2 + y^2 \right\}.$$

The map $u : \mathbb{R}^2 \to \mathcal{U}$, $u : (x, y) \mapsto (x, y, x^2 + y^2)$ is called *lifting map* onto the unit paraboloid.

For $p \in \mathbb{R}^2$ let H_p the plane tangent to $\mathcal{U}$ in $u(p)$, and $h_p : \mathbb{R}^3 \to H_p$ the projection of the x/y-plane onto H_p along the z-axis.

Lemma 11
$\|u(q) - h_p(q)\| = \|p - q\|^2$, for any $p, q \in \mathbb{R}^2$.

$\to$ Exercise

Theorem 12
Let $\mathcal{H}(P) := \bigcap_{p \in P} H_p^+$, where H_p^+ is the closed halfspace above H_p. The orthogonal projection of $\mathcal{H}(P)$ onto the x/y-plane is $VD(P)$.

Point Location

Theorem 13 (Kirkpatrick 1983) Let T be any triangulation of a set $P \subset \mathbb{R}^2$ of $n \geq 3$ points. Then in $\mathcal{O}(n)$ time and using $\mathcal{O}(n)$ space one can construct a data structure that finds in $\mathcal{O}(\log n)$ time for any query point $q \in \text{conv}(P)$ the triangle from T containing q.

Corollary 14 (Nearest Neighbor Search) For any set $P \subset \mathbb{R}^2$ of n points one can construct in

$\mathcal{O}(n \log n)$ preprocessing time and

using $\mathcal{O}(n)$ space

a data structure that finds in

$\mathcal{O}(\log n)$ query time

for any query point $q \in \text{conv}(P)$ the points closest to q among the points from P.

Kirkpatrick's Hierarchy

Plan: Construct a hierarchy $T_0, \ldots, T_h$ of triangulations, such that

1. $V(T_i) \subset V(T_{i-1})$, $i = 1, \ldots, h$;
2. $T_0 = T$;
3. T_h is a single triangle.

Search$(x \in \mathbb{R}^2)$

1. For $i = h \ldots 0$: Find the triangle t_i of T_i that contains x.
2. return t_0.

To make the search efficient, we need

(C1) Every triangle in T_i intersects few (at most c) triangles of T_{i-1}.

(C2) h is small ($\leq d \log n$).

Proposition 15 The search needs at most $3cd \log n = \mathcal{O}(\log n)$ orientation tests.

Thinning a triangulation

Observation 16 Removing a vertex v and all incident edges from a triangulation leaves a polygonal hole that is star-shaped (all points visible from v).

Lemma 17 A star-shaped polygon, given as a sequences of n vertices, can be triangulated in $\mathcal{O}(n)$ time. $\rightarrow$ Exercise.

Idea: Obtain T_i from T_{i-1} by removing a set I of independent vertices and re-triangulate.

The vertices in I should

a) have small degree (otw re-triangulation is too expensive) and

b) there should be many (otw the hierarchy gets too high).

Lemma 18 In any triangulation of $n \geq 3$ points in $\mathbb{R}^2$ one can find in $\mathcal{O}(n)$ time an independent set of $\geq n/18$ vertices of degree ≤ 8.

Proof. (of Theorem 13)
Construct $T_0, \ldots T_h$ with $T_0 = T$. Obtain T_i from T_{i-1} by removing an independent set U and re-triangulate the resulting holes.

Lemma 17 and Lemma 18: every step is linear in $|V(T_i)|$. In total

$$\sum_{i=0}^{h} \alpha |V(T_i)| \leq \sum_{i=0}^{h} \alpha n (17/18)^i < 18\alpha n = \mathcal{O}(n),$$

for some $\alpha > 0$.

Similarly for space, as any triangle in T_i is linked to at most 8 triangles from T_{i+1}.

$h = \log_{18/17} n < 12.2 \log n$.

By Proposition 15 the search needs at most $3 \cdot 8 \cdot \log_{18/17} n < 292 \log n$ orientation tests. □

Improvements

The constant 292 in the search time is not optimal.

- Sarnak, Tarjan (1986): $4 \log n$.
- Edelsbrunner, Guibas, Stolfi (1986): $3 \log n$.
- Goodrich, Orletsky, Ramaier (1997): $2 \log n$.
- Adamy, Seidel (2000): $1 \log n + 2\sqrt{\log n}$.