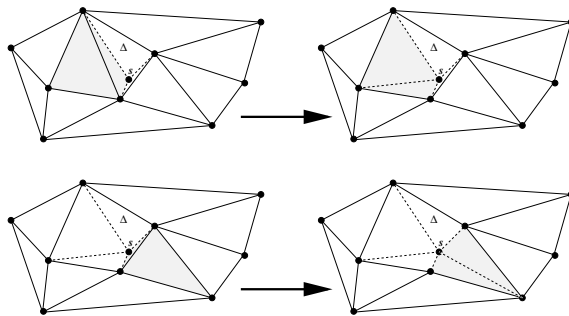


## Problem & Algorithm

Given a set  $P$  of  $n$  points in  $\mathbb{R}^2$  in general position (no 3 on a line, no 4 on a circle), compute the Delaunay triangulation  $\mathcal{DT}(P)$  of  $P$ .



**Randomized** Incremental Construction (RIC) of the Delaunay triangulation

**Algorithm:** *Randomized Incremental Construction*; points are inserted in random order (after the first three far-away points  $a, b, c$ )

**Def:**  $\mathcal{T}_r$  is the Delaunay triangulation after  $r$  insertion steps (a random variable).

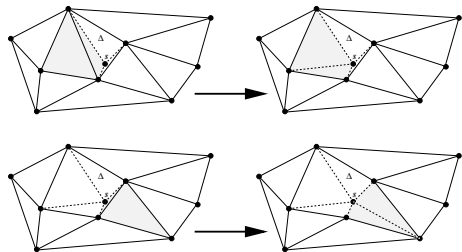
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$\mathcal{T}_{r-1} \rightarrow \mathcal{T}_r$ : Expected Update Cost

- Update cost =  $O(1) + O(\text{number of flips})$
- each flip generates a new edge of  $\mathcal{T}_r$  incident to the new point  $s$
- Update cost =

$O(\text{degree of } s \text{ in } \mathcal{T}_r)$



Example:  $s$  has degree 5 in  $\mathcal{T}_r$

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Backwards analysis

Run the algorithm backwards (as a movie).

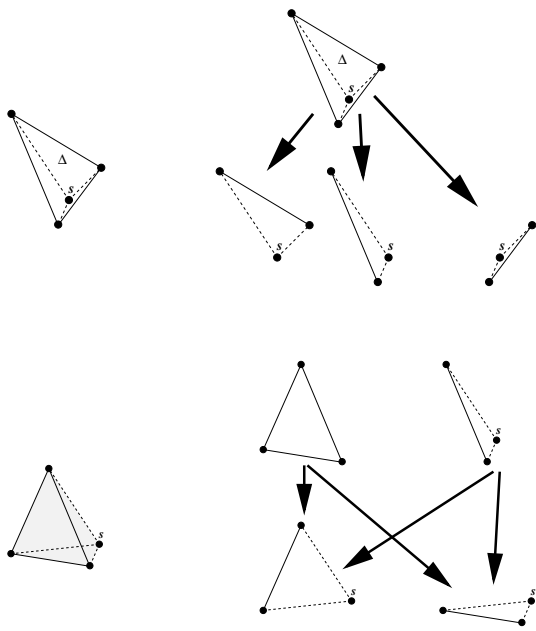
- Random insertion order  $\approx$  random deletion order
- In going from  $\mathcal{T}_r$  to  $\mathcal{T}_{r-1}$ , a *random* point  $s \neq a, b, c$  is being deleted
- Its average degree in  $\mathcal{T}_r$  is at most

$$\begin{aligned} & \frac{1}{r} \sum_{q \in \mathcal{T}_r \setminus \{a, b, c\}} \text{deg}(q, \mathcal{T}_r) \\ & \leq \frac{1}{r} \sum_{q \in \mathcal{T}_r} \text{deg}(q, \mathcal{T}_r) \\ & \leq \frac{1}{r} (2(3(r+3) - 6)) \approx 6. \end{aligned}$$

$\Rightarrow$  Overall expected update cost is  $O(1)$  per step and  $O(n)$  in total.

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Recall the *History graph* approach



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- Every triangle  $\Delta$  being traversed while locating  $s$  in  $\mathcal{T}_{r-1}$ 
  - is in  $\mathcal{T}_k$  for some  $k < r$
  - has the point  $s$  inside and in particular inside its circumcircle
- Such a pair  $(\Delta, s)$  is called a *conflict*.
- Expected Cost of all find steps =  $O(\text{expected total number of conflicts})$
- $D(\Delta)$ : the three vertices of  $\Delta$
- $K(\Delta)$ : the points in  $\Delta$ 's circumcircle
- $x_1, \dots, x_n$  the insertion order
- $\mathcal{T}(R) = \mathcal{DT}(R \cup \{a, b, c\})$

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### Expected number of conflicts

We want to count the total number of conflicts  $(s, \Delta)$ , i.e.

$$\sum_{r=0}^n \sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} |K(\Delta)|.$$

The following are equal: the triangles

- appearing in the step  $\mathcal{T}_{r-1} \rightarrow \mathcal{T}_r$ ,
- being in  $\mathcal{T}_r$  with  $x_r \in D(\Delta)$ .

For fixed  $R = \{x_1, \dots, x_r\}$ ,  $\text{prob}(x = x_r) = 1/r$  for  $x \in R$ , so the expected conflict number is

$$\begin{aligned} & \frac{1}{r} \sum_{x \in R} \sum_{\Delta \in \mathcal{T}(R), x \in D(\Delta)} \sum_{y \in X \setminus R} [y \in K(\Delta)] \\ & \leq \frac{3}{r} \sum_{y \in X \setminus R} |\{\Delta \in \mathcal{T}(R) \mid y \in K(\Delta)\}|. \end{aligned}$$

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### An easy but crucial Lemma

**Lemma.**

$$|\{\Delta \in \mathcal{T}(R) \mid y \in K(\Delta)\}|$$

=

$$|\mathcal{T}(R)| - |\mathcal{T}(R \cup \{y\})| + \text{deg}(y, \mathcal{T}(R \cup \{y\})).$$

**Proof.** The triangles of  $\mathcal{T}(R)$  not in conflict with  $y$  are exactly the triangles of  $\mathcal{T}(R \cup \{y\})$  that do not have  $y$  as a vertex.

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## Expected number of conflicts (II)

$K_r$ : expected number of new conflicts when  $x_r$  is inserted. Since  $R$  is random itself,  $K_r$  is bounded by

$$\frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{3}{r} \sum_{y \in X \setminus R} |\{\Delta \in \mathcal{T}(R) \mid y \in K(\Delta)\}|$$

which is

$$\underbrace{\frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{3}{r} \sum_{y \in X \setminus R} |\mathcal{T}(R)|}_{k_1} -$$

$$\underbrace{\frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{3}{r} \sum_{y \in X \setminus R} |\mathcal{T}(R \cup \{y\})|}_{k_2} +$$

$$\underbrace{\frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{3}{r} \sum_{y \in X \setminus R} \deg(y, \mathcal{T}(R \cup \{y\}))}_{k_3}.$$

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## Evaluating $k_1$

$$\begin{aligned} k_1 &= \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{3}{r} \sum_{y \in X \setminus R} |\mathcal{T}(R)| \\ &= \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} |\mathcal{T}(R)| \frac{3}{r} \sum_{y \in X \setminus R} 1 \\ &= \frac{3}{r} (n-r) t_r, \end{aligned}$$

where  $t_r$  is the expected number of triangles in  $\mathcal{T}_r$ .

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## Evaluating $k_2$

$$\begin{aligned} k_2 &= \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{3}{r} \sum_{y \in X \setminus R} |\mathcal{T}(R \cup \{y\})| \\ &= \frac{1}{\binom{n}{r}} \sum_{R' \subseteq X, |R'|=r+1} \frac{3}{r} \sum_{y \in R'} |\mathcal{T}(R')| \\ &= \frac{1}{\binom{n}{r+1}} \sum_{R' \subseteq X, |R'|=r+1} \frac{\binom{r+1}{r}}{\binom{n}{r}} \frac{3}{r} (r+1) |\mathcal{T}(R')| \\ &= \frac{1}{\binom{n}{r+1}} \sum_{R' \subseteq X, |R'|=r+1} \frac{3}{r} (n-r) |\mathcal{T}(R')| \\ &= \frac{3}{r} (n-r) t_{r+1} \\ &= \frac{3}{r+1} (n - (r+1)) t_{r+1} + \frac{3n}{r(r+1)} t_{r+1}. \end{aligned}$$

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## Evaluating $k_3$

$$\begin{aligned} k_3 &= \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{3}{r} \sum_{y \in X \setminus R} \deg(y, \mathcal{T}(R \cup \{y\})) \\ &= \frac{1}{\binom{n}{r}} \sum_{R' \subseteq X, |R'|=r+1} \frac{3}{r} \sum_{y \in R'} \deg(y, \mathcal{T}(R')) \\ &\leq \frac{1}{\binom{n}{r}} \sum_{R' \subseteq X, |R'|=r+1} \frac{3}{r} 3 |\mathcal{T}(R')| \\ &= \frac{1}{\binom{n}{r+1}} \sum_{R' \subseteq X, |R'|=r+1} \frac{\binom{r+1}{r}}{\binom{n}{r}} \frac{3}{r} 3 |\mathcal{T}(R')| \\ &= \frac{1}{\binom{n}{r+1}} \sum_{R' \subseteq X, |R'|=r+1} \frac{n-r}{r+1} \cdot \frac{3}{r} 3 |\mathcal{T}(R')| \\ &= \frac{3^2}{r(r+1)} (n-r) t_{r+1} \\ &= \frac{3^2 n}{r(r+1)} t_{r+1} - \frac{3^2}{r+1} t_{r+1}. \end{aligned}$$

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## Expected number of conflicts (III)

In step  $n$ , no conflict is created. In step 0, it's  $n$  conflicts. Moreover,  $k_1(r+1)$  cancels with the first term of  $k_2(r)$ , and we get

$$\begin{aligned} \sum_{r=1}^{n-1} K_r &\leq \sum_{r=1}^{n-1} (k_1 - k_2 + k_3) \\ &\leq 3(n-1)t_1 + \\ &\quad 3(3+1)n \sum_{r=1}^{n-1} \frac{t_{r+1}}{r(r+1)} - \\ &\quad 3^2 \sum_{r=1}^{n-1} \frac{t_{r+1}}{r+1}. \end{aligned}$$

## Back to Delaunay triangulations

- $t_r \leq 2r - 4 = O(r)$
- $\sum_{r=1}^{n-1} K_r = O(n + nH_{n-1}) \Rightarrow O(n \log n)$ .

**Theorem:** The Delaunay triangulation of  $n$  points in general position in  $\mathbb{R}^2$  can be computed in expected time

$$O(n \log n).$$

**Remark:** The same analysis can be done for other problems (get to at least one more later).