Randomized Incremental Construction (RIC) of the Delaunay triangulation

**Problem & Algorithm**

Given a set $P$ of $n$ points in $\mathbb{R}^2$ in general position (no 3 on a line, no 4 on a circle), compute the Delaunay triangulation $DT(P)$ of $P$.

**Algorithm:** Randomized Incremental Construction; points are inserted in random order (after the first three far-away points $a, b, c$)

**Def:** $T_r$ is the Delaunay triangulation after $r$ insertion steps (a random variable).

$\mathcal{T}_{r-1} \rightarrow \mathcal{T}_r$: Expected Update Cost

- Update cost $= O(1) + O(\text{number of flips})$
- each flip generates a new edge of $\mathcal{T}_r$ incident to the new point $s$
- Update cost $= O(\text{degree of } s \text{ in } \mathcal{T}_r)$

Example: $s$ has degree 5 in $\mathcal{T}_r$

Backwards analysis

Run the algorithm backwards (as a movie).

- Random insertion order $\approx$ random deletion order
- In going from $\mathcal{T}_r$ to $\mathcal{T}_{r-1}$, a random point $s \neq a, b, c$ is being deleted
- Its average degree in $\mathcal{T}_r$ is at most
  \[
  \frac{1}{r} \sum_{q \in \mathcal{T}_r \setminus \{a, b, c\}} \deg(q, \mathcal{T}_r) 
  \leq \frac{1}{r} \sum_{q \in \mathcal{T}_r} \deg(q, \mathcal{T}_r) 
  \leq \frac{1}{r} (2(3(r + 3) - 6)) \approx 6.
  \]

$\Rightarrow$ Overall expected update cost is $O(1)$ per step and $O(n)$ in total.
\( T_{r-1} \rightarrow T_r: \) Expected Find Cost

Recall the \textit{History graph} approach

\begin{equation}
\sum_{r=0}^{n} \sum_{\Delta \in T_r \setminus T_{r-1}} |K(\Delta)|.
\end{equation}

The following are equal: the triangles

- appearing in the step \( T_{r-1} \rightarrow T_r \),
- being in \( T_r \) with \( x_r \in D(\Delta) \).

For fixed \( R = \{x_1, \ldots, x_r\} \), \( \text{prob}(x = x_r) = 1/r \) for \( x \in R \), so the expected conflict number is

\begin{align*}
\frac{1}{r} \sum_{x \in R} \sum_{\Delta \in T(R), x \in D(\Delta)} \sum_{y \in X \setminus R} [y \in K(\Delta)]
\leq \frac{3}{r} \sum_{y \in X \setminus R} |\{\Delta \in T(R) \mid y \in K(\Delta)\}|.
\end{align*}

\( T_{r-1} \rightarrow T_r: \) Expected Find Cost

- Every triangle \( \Delta \) being traversed while locating \( s \) in \( T_{r-1} \)
  - is in \( T_k \) for some \( k < r \)
  - has the point \( s \) inside and in particular inside its circumcircle
- Such a pair \((\Delta, s)\) is called a \textit{conflict}.
- Expected Cost of all find steps = \( O(\text{expected total number of conflicts}) \)
- \( D(\Delta) \): the three vertices of \( \Delta \)
- \( K(\Delta) \): the points in \( \Delta \)'s circumcircle
- \( x_1, \ldots, x_n \) the insertion order
- \( T(R) = DT(R \cup \{a, b, c\}) \)

Expected number of conflicts

We want to count the total number of conflicts \((s, \Delta)\), i.e.

\begin{equation}
\sum_{r=0}^{n} \sum_{\Delta \in T_r \setminus T_{r-1}} |K(\Delta)|.
\end{equation}

The following are equal: the triangles

- appearing in the step \( T_{r-1} \rightarrow T_r \),
- being in \( T_r \) with \( x_r \in D(\Delta) \).

For fixed \( R = \{x_1, \ldots, x_r\} \), \( \text{prob}(x = x_r) = 1/r \) for \( x \in R \), so the expected conflict number is

\begin{align*}
\frac{1}{r} \sum_{x \in R} \sum_{\Delta \in T(R), x \in D(\Delta)} \sum_{y \in X \setminus R} [y \in K(\Delta)]
\leq \frac{3}{r} \sum_{y \in X \setminus R} |\{\Delta \in T(R) \mid y \in K(\Delta)\}|.
\end{align*}

An easy but crucial Lemma

\textbf{Lemma.}

\begin{align*}
|\{\Delta \in T(R) \mid y \in K(\Delta)\}| &= |T(R)| - |T(R \cup \{y\})| + \text{deg}(y, T(R \cup \{y\})).
\end{align*}

\textbf{Proof.} The triangles of \( T(R) \) not in conflict with \( y \) are exactly the triangles of \( T(R \cup \{y\}) \) that do not have \( y \) as a vertex.
Expected number of conflicts (II)

\( K_r \): expected number of new conflicts when \( x_r \) is inserted. Since \( R \) is random itself, \( K_r \) is bounded by

\[
\frac{1}{\binom{n}{r}} \sum \frac{1}{R \subseteq X, |R| = r} \sum \frac{3}{y \in X \setminus R} \; |\{ \Delta \in T(R) \mid y \in K(\Delta) \}|
\]

which is

\[
\frac{1}{\binom{n}{r}} \sum \frac{1}{R \subseteq X, |R| = r} \sum \frac{3}{y \in X \setminus R} |T(R)| - k_2
\]

\[
\frac{1}{\binom{n}{r}} \sum \frac{3}{R' \subseteq X, |R'| = r+1} \sum \frac{3}{y \in R'} |T(R')| + k_2
\]

\[
\frac{1}{\binom{n}{r}} \sum \frac{3}{R' \subseteq X, |R'| = r+1} \sum \frac{3}{y \in R'} \deg(y, T(R)) \cdot k_3
\]

**Evaluating \( k_1 \)**

\[
k_1 = \frac{1}{\binom{n}{r}} \sum \frac{3}{R \subseteq X, |R| = r} \sum \frac{3}{y \in X \setminus R} |T(R)|
\]

\[
= \frac{1}{\binom{n}{r}} \sum |T(R)| \sum \frac{3}{y \in X \setminus R} 1
\]

\[
= \frac{3}{r} (n - r)t_r,
\]

where \( t_r \) is the expected number of triangles in \( T_r \).

**Evaluating \( k_2 \)**

\[
k_2 = \frac{1}{\binom{n}{r}} \sum \frac{3}{R \subseteq X, |R| = r} \sum \frac{3}{y \in X \setminus R} |T(R)\cup\{y\}|
\]

\[
= \frac{1}{\binom{n}{r}} \sum \frac{3}{R' \subseteq X, |R'| = r+1} \sum \frac{3}{y \in R'} |T(R')|
\]

\[
= \frac{1}{\binom{n}{r+1}} \sum \frac{3}{R' \subseteq X, |R'| = r+1} \sum \frac{3}{n-r} |T(R')|
\]

\[
= \frac{3}{r} (n - r)t_{r+1}
\]

\[
= \frac{3}{r+1} (n - (r + 1))t_{r+1} + \frac{3n}{r(r+1)} t_{r+1}.
\]

**Evaluating \( k_3 \)**

\[
k_3 = \frac{1}{\binom{n}{r}} \sum \frac{3}{R \subseteq X, |R| = r} \sum \frac{3}{y \in X \setminus R} \deg(y, T(R)\cup\{y\})
\]

\[
= \frac{1}{\binom{n}{r}} \sum \frac{3}{R' \subseteq X, |R'| = r+1} \sum \frac{3}{y \in R'} \deg(y, T(R'))
\]

\[
\leq \frac{1}{\binom{n}{r}} \sum \frac{3}{R' \subseteq X, |R'| = r+1} |T(R')|
\]

\[
= \frac{1}{\binom{n}{r+1}} \sum \frac{3}{R' \subseteq X, |R'| = r+1} \frac{n-r}{r+1} |T(R')|
\]

\[
= \frac{3^2}{r(r+1)} (n - r)t_{r+1}
\]

\[
= \frac{3^2n}{r(r+1)} - \frac{3^2}{r+1} t_{r+1}.
\]
Expected number of conflicts (III)

In step $n$, no conflict is created. In step 0, it’s $n$ conflicts. Moreover, $k_1(r+1)$ cancels with the first term of $k_2(r)$, and we get

$$
\sum_{r=1}^{n-1} K_r \leq \sum_{r=1}^{n-1} (k_1 - k_2 + k_3) \\
\leq 3(n-1)t_1 + \\
3(3+1)n \sum_{r=1}^{n-1} \frac{t_{r+1}}{r(r+1)} - \\
32 \sum_{r=1}^{n-1} \frac{t_{r+1}}{r+1}.
$$

Back to Delaunay triangulations

- $t_r \leq 2r - 4 = O(r)$
- $\sum_{r=1}^{n-1} K_r = O(n + nH_{n-1}) \Rightarrow O(n \log n)$.

**Theorem:** The Delaunay triangulation of $n$ points in general position in $\mathbb{R}^2$ can be computed in expected time $O(n \log n)$.

**Remark:** The same analysis can be done for other problems (get to at least one more later).