1. Convex Hull

Lecture on Monday 21st September, 2009 by Michael Hoffmann <hoffmann@inf.ethz.ch>

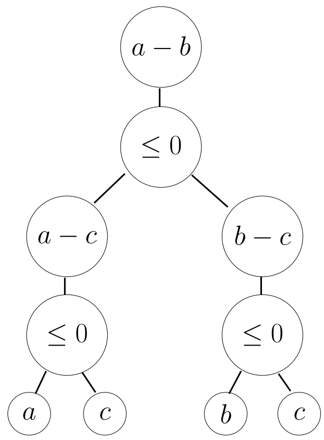
1.1 Models of Computation

Real RAM Model. A memory cell stores a real number. Any single arithmetic operation or comparison can be computed in constant time. In addition, sometimes also roots, logarithms, other analytic functions, indirect addressing (integral), or floor and ceiling are used.

This is a quite powerful (and somewhat unrealistic) model of computation. Therefore we have to ensure that we do not abuse its power.

Algebraic Computation Trees (Ben-Or [1]). A computation is regarded a binary tree.

- The leaves contain the (possible) results of the computation.
- Every node v with one child an operation of the form $+, -, *, /, \sqrt{}, \ldots$ is associated to. The operands of this operation are constant, input values, or among the ancestors of v in the tree.
- Every node v with two children a branching of the form > 0, ≥ 0, or $= 0$ is associated to. The branch is with respect to the result of v's parent node. If the expression yields `true`, the computation continues with the left child of v; otherwise, it continues with the right child of v.



If every branch is based on a linear function in the input values, we face a *linear computation tree*. Analogously one can define, say, quadratic computation trees.

The term *decision tree* is used if all of the results are either `true` or `false`.

1.2 Basic Geometric Objects

We will mostly be concerned with the d-dimensional Euclidean space $\mathbb{R}^d$, for small $d \in \mathbb{N}$; typically, $d = 2$ or $d = 3$. The basic objects of interest in $\mathbb{R}^d$ are the following.

Points. A point p, typically described by its d Cartesian coordinates $p = (x_1, \ldots, x_d)$.

Directions. A vector $v \in \mathcal{S}^{d-1}$ (the $(d-1)$-dimensional unit sphere), typically described by its d Cartesian coordinates $v = (x_1, \ldots, x_d)$.

Lines. A line is a one-dimensional affine subspace. It can be described by a point p and a direction d as the set of all points r that satisfy $r = p + \lambda d$, for some $\lambda \in \mathbb{R}$.

Rays. A ray is a connected component of what remains if one removes a single point from a line. It can be described by a point p and a direction d as the set of all points r that satisfy $r = p + \lambda d$, for some $\lambda \geq 0$.

Line segment. A line segment is the bounded connected component of what remains if one removes a single point from a ray. It can be described by two points p and q as the set of all points r that satisfy $r = p + \lambda(q - p)$, for some $\lambda \in [0, 1]$. We will denote the line segment through p and q by $\overline{pq}$.

Hyperplanes. A hyperplane $\mathcal{H}$ is a $(d-1)$-dimensional affine subspace. It can be described algebraically by $d + 1$ coefficients $\lambda_1, \ldots, \lambda_{d+1} \in \mathbb{R}$ as the set of all points $(x_1, \ldots, x_d)$ that satisfy the linear equation $\mathcal{H} : \sum_{i=1}^{d} \lambda_i x_i = \lambda_{d+1}$.

Spheres. A sphere is the set of all points that are equidistant to a fixed point. It can be described by a point c (center) and a number $\rho \in \mathbb{R}$ (radius) as the set of all points p that satisfy $||p - c|| \leq \rho$.

1.3 Convexity

Consider $P \subset \mathbb{R}^d$. The following terminology should be familiar from linear algebra courses.

Linear hull.

$$\text{lin}(P) := \left\{ q \;\middle|\; q = \sum \lambda_i p_i \wedge \forall\, i\, :\, p_i \in P,\, \lambda_i \in \mathbb{R} \right\}$$

(smallest linear subspace containing P). For instance, if $P = \{p\} \subset \mathbb{R}^2 \setminus \{0\}$ then $\text{lin}(P)$ is the line through p and the origin.

Affine hull.

$$\text{aff}(P) := \left\{ q \;\middle|\; q = \sum \lambda_i p_i \wedge \sum \lambda_i = 1 \wedge \forall\, i\, :\, p_i \in P,\, \lambda_i \in \mathbb{R} \right\}$$

(smallest affine subspace containing P). For instance, if $P = \{p, q\} \subset \mathbb{R}^2$ and $p \neq q$ then $\text{aff}(P)$ is the line through p and q.

Convex hull.

Definition 1.1 *A set* $P \subseteq \mathbb{R}^d$ *is* **convex** *if and only if* $\overline{pq} \subseteq P$*, for any* $p, q \in P$.

Theorem 1.2 *A set* $P \subseteq \mathbb{R}^d$ *is convex if and only if* $\sum_{i=1}^{n} \lambda_i p_i \in P$*, for all* $n \in \mathbb{N}$, $p_1, \ldots, p_n \in P$*, and* $\lambda_1, \ldots, \lambda_n \geq 0$ *with* $\sum_{i=1}^{n} \lambda_i = 1$.

Proof. "$\Leftarrow$": obvious with $n = 2$.

"$\Rightarrow$": Induction on n. For $n = 1$ the statement is trivial. For $n \geq 2$, let $p_i \in P$ and $\lambda_i \geq 0$, for $1 \leq i \leq n$, and assume $\sum_{i=1}^{n} \lambda_i = 1$. We may suppose that $\lambda_i > 0$, for all i. (Simply omit those points whose coefficient is zero.)

Define $\lambda = \sum_{i=1}^{n-1} \lambda_i$ and for $1 \leq i \leq n-1$ set $\mu_i = \lambda_i/\lambda$. Observe that $\mu_i \geq 0$ and $\sum_{i=1}^{n-1} \mu_i = 1$. By the inductive hypothesis, $q := \sum_{i=1}^{n-1} \mu_i p_i \in P$, and thus by convexity of P also $\lambda q + (1-\lambda)p_k \in P$. We conclude by noting that $\lambda q + (1-\lambda)p_k = \lambda \sum_{i=1}^{n-1} \mu_i p_i + \lambda_k p_k = \sum_{i=1}^{n} \lambda_i p_i$. □

Observation 1.3 *For any family* $(P_i)_{i \in I}$ *of convex sets the intersection* $\bigcap_{i \in I} P_i$ *is convex.*

Definition 1.4 *The* **convex hull** $\text{conv}(P)$ *of a set* $P \subset \mathbb{R}^d$ *is the intersection of all convex supersets of* P.

By Observation 1.3, the convex hull is convex, indeed.

Theorem 1.5 *For any* $P \subseteq \mathbb{R}^d$ *we have*

$$\text{conv}(P) = \left\{ \sum_{i=1}^{n} \lambda_i p_i \,\middle|\, n \in \mathbb{N} \wedge \sum_{i=1}^{n} \lambda_i = 1 \wedge \forall i \in \{1, \ldots, n\} : \lambda_i \geq 0 \wedge p_i \in P \right\}.$$

Proof. "$\supseteq$": Consider a convex set $C \supseteq P$. By Theorem 1.2 the right hand side is contained in C. As C was arbitrary, the claim follows.

"$\subseteq$": We show that the right hand side forms a convex set. Let $p = \sum_{i=1}^{n} \lambda_i p_i$ and $q = \sum_{i=1}^{n} \mu_i p_i$ be two convex combinations. (We may suppose that both p and q are expressed over the same p_i by possibly adding some terms with a coefficient of zero.)

Then for $\lambda \in [0, 1]$ we have $\lambda p + (1-\lambda)q = \sum_{i=1}^{n} (\lambda\lambda_i + (1-\lambda)\mu_i)p_i \in P$, as $\sum_{i=1}^{n} (\lambda\lambda_i + (1-\lambda)\mu_i) = \lambda + (1-\lambda) = 1$. □

Definition 1.6 *The convex hull of a finite point set* $P \subset \mathbb{R}^d$ *forms a* **convex polytope***. Each* $p \in P$ *for which* $p \notin \text{conv}(P \setminus \{p\})$ *is called a* **vertex** *of* $\text{conv}(P)$*. A vertex of* $\text{conv}(P)$ *is also called an* **extremal point** *of* P.

Essentially, the following theorem shows that the term vertex above is well defined.

Theorem 1.7 *A convex polytope in* $\mathbb{R}^d$ *is the convex hull of its vertices.*

Proof. Let $P = \mathrm{conv}(p_1, \ldots, p_n)$, $n \in \mathbb{N}$, such that without loss of generality $p_1, \ldots, p_k$ are the vertices of $\mathcal{P} := \mathrm{conv}(P)$. We prove by induction on n that $\mathrm{conv}(p_1, \ldots, p_n) \subseteq \mathrm{conv}(p_1, \ldots, p_k)$. For $n = k$ the statement is trivial.

For $n > k$, p_n is not a vertex of $\mathcal{P}$ and hence p_n can be expressed as a convex combination $p_n = \sum_{i=1}^{n-1} \lambda_i p_i$. Thus for any $x \in \mathcal{P}$ we can write $x = \sum_{i=1}^{n} \mu_i p_i = \sum_{i=1}^{n-1} \mu_i p_i + \mu_k \sum_{i=1}^{n-1} \lambda_i p_i = \sum_{i=1}^{n-1} (\mu_i + \mu_k \lambda_i) p_i$. As $\sum_{i=1}^{n-1} (\mu_i + \mu_k \lambda_i) = 1$, we conclude by the inductive hypothesis that $x \in \mathrm{conv}(p_1, \ldots, p_k)$. □

Theorem 1.8 (Carathéodory [2]) *For any* $P \subset \mathbb{R}^d$ *and* $q \in \mathrm{conv}(P)$ *there exist* $k \leq d + 1$ *points* $p_1, \ldots, p_k \in P$ *such that* $q \in \mathrm{conv}(p_1, \ldots, p_k)$.

Theorem 1.9 (Separation Theorem) *Any two compact convex sets* $C, D \subset \mathbb{R}^d$ *with* $C \cap D = \emptyset$ *can be separated strictly by a hyperplane, that is, there exists a hyperplane* h *such that* C *and* D *lie in the opposite open halfspaces bounded by* h.

Proof. Consider the distance function $d : C \times D \to \mathbb{R}$ with $(c, d) \mapsto ||c - d||$. Since $C \times D$ is compact and d is continuous and strictly bounded from below by 0, d attains its minimum at some point $(c_0, d_0) \in C \times D$. Let h be the hyperplane perpendicular to the line segment $\overline{c_0 d_0}$ and passing through the midpoint of c_0 and d_0.

If there was a point, say, c' in $C \cap h$, then by convexity of C the whole line segment $c_o c'$ lies in C and some point along this segment is closer to d_0 than is c_0, in contradiction to the choice of c_0. If, say, C has points on both sides of h, then by convexity of C it has also a point on h, but we just saw that there is no such point. Therefore, C and D must lie in different open halfspaces bounded by h. □

Actually, the statement above holds for arbitrary (not necessarily compact) convex sets, but the separation is not necessarily strict (the hyperplane may have to intersect the sets) and the proof is a bit more involved (cf. [4], but also check the errata on Jirka's webpage).

Altogether we obtain various equivalent definitions for the convex hull, summarized in the following theorem.

Theorem 1.10 *For a compact set* $P \subset \mathbb{R}^d$ *we can characterize* $\mathrm{conv}(P)$ *equivalently as one of*

(a) the smallest convex subset of $\mathbb{R}^d$ *that contains* P;

(b) the set of all convex combinations of points from P;

(c) the set of all convex combinations formed by $d + 1$ *or fewer points from* P;

(d) the intersection of all convex supersets of P;

(e) the intersection of all closed halfspaces containing P.

Note that compactness is needed for (d) $\Longleftrightarrow$ (e) only.

1.4 The convex hull problem in $\mathbb{R}^2$

Convex hull

Input: $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$, $n \in \mathbb{N}$.

Output: Sequence $(q_1, \ldots, q_h)$, $1 \leq h \leq n$, of the vertices of $\operatorname{conv}(P)$ (ordered counterclockwise).

Extremal points

Input: $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$, $n \in \mathbb{N}$.

Output: Set $Q \subseteq P$ of the vertices of $\operatorname{conv}(P)$.

Degeneracies. Three points collinear. Which are extremal?

By the Separation Theorem, every extremal point p can be separated from the convex hull of the remaining points by a halfplane. If we take such a halfplane and shift its defining line such that it passes through p, then all points from P other than p should lie in the resulting open halfplane.

The following definition is chosen such that among collinear points the interior ones are not considered extremal.

Definition 1.11 *A point $p \in P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$ is* **extremal** *for P $\iff$ there is a directed line g through p such that $P \setminus \{p\}$ is to the left of g.*

1.5 Trivial algorithms

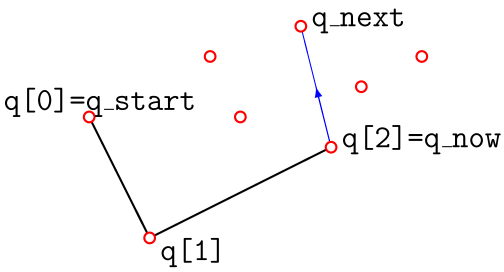
One can compute the extremal points using Carathéodory's Theorem as follows: Test for every point $p \in P$ whether there are $q, r, s \in P \setminus \{p\}$ such that p is inside the triangle with vertices q, r, and s. Runtime $O(n^4)$.

Another option, inspired by the Separation Theorem: test for every pair $(p, q) \in P^2$ whether all points from $P \setminus \{p, q\}$ are to the left of the directed line through p and q (or on the line segment $\overline{pq}$). Runtime $O(n^3)$.

1.6 Jarvis' Wrap

Find a point p_1 that is a vertex of $\operatorname{conv}(P)$ (e.g., the one with smallest x-coordinate). "Wrap" P starting from p_1, i.e., always find the next vertex of $\operatorname{conv}(P)$ as the one that is rightmost with respect to the previous vertex.

Besides comparing x-coordinates, the only geometric primitive needed is an *orientation* test: Denote by $\texttt{rightturn}(p, q, r)$, for three points $p, q, r \in \mathbb{R}^2$, the predicate that is true if and only if r is (strictly) to the right of the oriented line pq.



Analysis. For every output point the above algorithm spends n rightturn tests, which is $\Rightarrow O(nh)$ in total.

Theorem 1.12 [3] *Jarvis' Wrap computes the convex hull of n points in $\mathbb{R}^2$ using $O(nh)$ rightturn tests, where h is the number of hull vertices.*

In the worst case we have $h = n$, that is, $O(n^2)$ rightturn tests. Jarvis' Wrap has a remarkable property that is called *output dependence*: the runtime depends not only on the size of the input but also on the size of the output. For a huge point set it constructs the convex hull in optimal linear time, if the convex hull consists of a constant number of vertices only. Unfortunately the worst case performance of Jarvis' Wrap is suboptimal, as we will see soon.

Degeneracies.

- Several points have smallest x-coordinate $\Rightarrow$ lexicographic order:

$$(p_x, p_y) < (q_x, q_y) \iff p_x < q_x \vee p_x = q_x \wedge p_y < q_y .$$

- Several points identical.
- Three or more points collinear $\Rightarrow$ choose the point that is farthest among those that are rightmost.

References

[1] M. Ben-Or, Lower bounds for algebraic computation trees, in: *Proc. 15th Annu. ACM Sympos. Theory Comput.*, 1983, 80–86.

[2] C. Carathéodory, Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen, *Rendiconto del Circolo Matematico di Palermo* **32** (1911), 193–217.

[3] R. A. Jarvis, On the identification of the convex hull of a finite set of points in the plane, *Inform. Process. Lett.* **2** (1973), 18–21.

[4] Jiří Matoušek, *Lectures on Discrete Geometry*, Springer, New York, NY, 2002.