Flow network (D, c, s, t)

directed graph $D = (V, E)$; $|V| = n$, $|E| = m$,
capacity $c : V \times V \rightarrow \mathbb{R}^+ \cup \{0\}$,
such that $c(uv) = 0$ if $uv \notin E$
source s, sink t

Notation: ordered pair (u, v) is denoted by uv.
Assumption: for every vertex $v \in V$ there is an st-path through v. Thus $m \geq n - 1$.

$f : V \times V \rightarrow \mathbb{R}$ is a flow on the network (D, c, s, t) if it satisfies

1. Capacity constraint:

$$f(uv) \leq c(uv) \text{ for every } u, v \in V$$

2. Skew symmetry:

$$f(uv) = -f(vu) \text{ for every } u, v \in V$$

3. Flow conservation:

$$\sum_{v \in V} f(uv) = 0 \text{ for every } u \in V \setminus \{s, t\}$$

The `MaxFlow` problem

value of flow f: $|f| := \sum_{v \in V} f(sv)$.

maximum flow of a network: a flow whose value is maximum over all flows of the network

The Problem
Given a flow network (D, c, s, t), find a maximum flow.

Further notation and basic properties

$$f(X,Y) := \sum_{x\in X}\sum_{y\in Y} f(xy).$$

Examples: 1. Flow conservation property:

$f(\{u\},V) = 0$ for every $u \in V \setminus \{s,t\}$.

2. Value of flow f: $|f| = f(\{s\},V)$

Lemma Let f be a skew-symmetric function. Then

(i) $f(X,X) = 0$ for all $X \subseteq V$

(ii) $f(X,Y) = -f(Y,X)$ for all $X,Y \subseteq V$

(iii) $f(X\cup Y,Z) = f(X,Z) + f(Y,Z)$ and
$f(Z,X\cup Y) = f(X,Z) + f(Y,Z)$
for all $X,Y,Z \subseteq V$ with $X\cap Y = \emptyset$

Claim $|f| = f(V,t)$

Proof: Use Lemma and flow conservation.

Residual network

residual network (D_f, c_f, s, t)

residual capacity $c_f(uv) = c(uv) - f(uv)$

residual digraph $D_f = (V, E_f)$

where $E_f = \{uv \in V \times V : c_f(uv) > 0\}$

Remark Edges in E_f are either edges in E or their reversals. Hence $|E_f| \leq 2m$.

Lemma Let f be a flow in the flow network D and let f' be a flow in the residual flow network D_f. Then $f + f'$ is a flow in D with value $|f| + |f'|$.

Augmenting paths

An st-path P in D_f is called an augmenting path.

residual capacity of P:

$$c_f(P) = \min\{c_f(uv) : uv \text{ is on } P\}$$

Define

$$f_P(uv) := \begin{cases} c_f(P) & \text{if } uv \text{ is on } P \\ -c_f(P) & \text{if } vu \text{ is on } P \\ 0 & \text{otherwise} \end{cases}$$

Claim f_P is a flow in D_f.

Corollary
$f + f_P$ is a flow in D with value larger than $|f|$.

Ford-Fulkerson Method

Inititalization $f \equiv 0$

WHILE there exists an augmenting path P

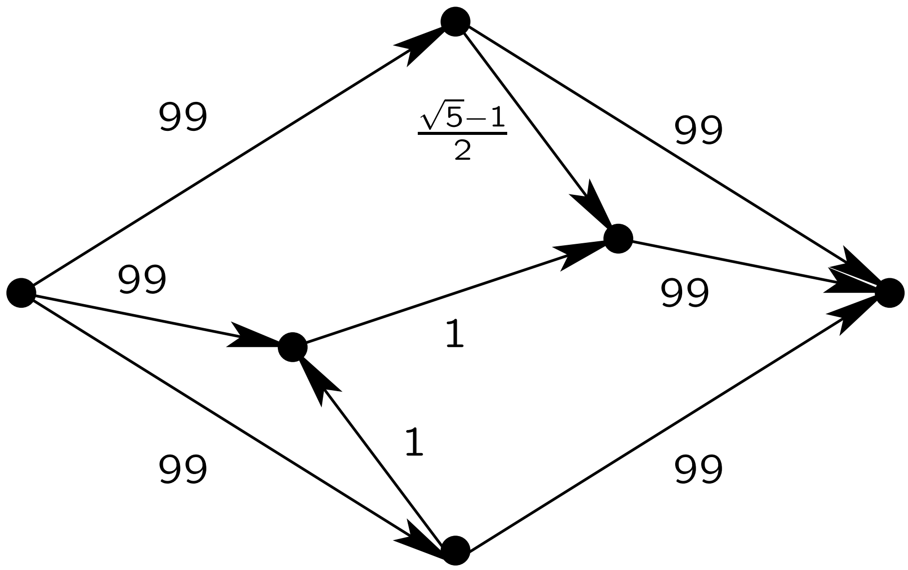
DO augment flow f along P

return f

Running times:

- Basic (careless) Ford-Fulkerson: might not even terminate, flow value might not converge to maximum;
  when capacities are integers, it terminates in time $O(m\,|f^*|)$, where f^* is a maximum flow.
- Edmonds-Karp: chooses a *shortest* augmenting path; runs in $O(nm^2)$

Example



Example of Zwick (1995)

Remark. The max flow is 199. There is such an unfortunate choice of a sequence of augmenting paths, by which the flow value tends to 3. Even our implementation can be cheated to do this by introducing an extra vertex in the middle of each edge.

Cuts of flow networks

cut $[S, T]$ of a flow network (D, c, s, t) is a partition of V into S and T such that $s \in S$ and $t \in T$

capacity of cut $[S, T]$: $c(S, T) = \sum_{x \in S} \sum_{y \in T} c(xy)$

minimum cut of a network: the one whose capacity is minimum over all cuts of the network.

Lemma $f(S, T) = |f|$ for any cut $[S, T]$.

Corollary For any flow f and any cut $[S, T]$

$$|f| \leq c(S, T).$$

Max-flow/min-cut Theorem The following are equivalent.

(i) f is a maximum flow

(ii) D_f contains no augmenting paths.

(iii) $|f| = c(S, T)$ for some cut $[S, T]$.

Preflow

f is a preflow if the following hold.

1. Capacity constraint:

$$f(uv) \leq c(uv) \text{ for every } u, v \in V$$

2. Skew symmetry:

$$f(uv) = -f(vu) \text{ for every } u, v \in V$$

3. Relaxed “flow conservation”:

$$f(V, u) = \sum_{v \in V} f(uv) \geq 0 \text{ for every } u \in V \setminus \{s\}$$

excess flow into u: $e(u) := f(V, u)$

vertex $u \in V \setminus \{s, t\}$ is overflowing if $e(u) > 0$

Lemma For any overflowing vertex u there is a us-path in the residual network D_f.
In particular, there is a residual outgoing edge from u.

Height function

$h : V \to \mathbb{N}$ is a height function for a preflow f if

- $h(s) = n$,
- $h(t) = 0$,
- $h(u) \leq h(v) + 1$ for every residual edge $uv \in E_f$

Lemma If f is a preflow which has a height function then there is no augmenting path in the residual network D_f.

Corollary If f is a flow which has a height function, then f is a maximum flow.

Lemma If f is a preflow with a height function h, then for any overflowing vertex u we have $h(u) \leq 2n - 1$.

Initialization

Every flow network (D, c, s, t) has a preflow with a height function:

INITIALIZE-PREFLOW(D, c, s, t)
FOR each pair $uv \in V \times V$
  DO $f(uv) := 0$
FOR each vertex $u \in N^{+}(s)$
  DO $f(su) := c(su)$
    $f(us) := -c(su)$
FOR each vertex $u \in V$
  DO $h(u) := 0$
$h(s) := n$

Claim INITIALIZE-PREFLOW outputs a preflow f of (D, c, s, t) with a height function h.

The GENERIC-PUSH-RELABEL algorithm maintains a preflow with a height function while performing a series of basic operations (PUSHes and RELABELs) and eventually outputing a flow with a height function.

The PUSH operation

PUSH(u, v) is applicable if

- u is overflowing,
- $c_f(uv) > 0$ (that is, $uv \in E_f$), and
- $h(u) = h(v) + 1$

Action: $d_f(uv) := \min\{e(u), c_f(uv)\}$ amount of flow is "pushed from u to v":

$$f(uv) := f(uv) + d_f(uv)$$
$$f(vu) := -f(uv)$$

Remark: Preflow changes, height function does not.

1. saturating push: if $d_f(uv) = c_f(uv)$.

After a saturating push uv becomes "saturated", i.e., $c_f(uv)$ becomes 0.

2. nonsaturating push: if $d_f(uv) = e(v)$.

After a nonsaturating push u is no longer overflowing.

The RELABEL operation

RELABEL(u) applies if

- u is overflowing and
- $h(u) \leq h(v)$ for all residual edges $uv \in E_f$.

Action: Define new height for u

$$h(u) := 1 + \min\{h(v) : uv \in E_f\}$$

Remark: Minimum is well-defined, because u is overflowing, so *there is* an outgoing residual edge.

Remark: Height function changes, preflow does not.

Remark: s and t cannot be relabeled

The GENERIC-PUSH-RELABEL Algorithm

INITIALIZE-PREFLOW(D, c, s, t)

WHILE there exists an applicable push or relabel operation
    DO select an applicable push or relabel operation
        and perform it

Correctness of the push-relabel method

Theorem If `GENERIC-PUSH-RELABEL` algorithm terminates then the preflow it computes is a maximum flow of the network D.

Proof:

Lemma If u is an overflowing vertex then either a push or a relabel operation applies to it.

Corollary At termination f is a flow.

Lemma h is maintained as a height function.

Proof:

Lemma During execution $h(u)$ never decreases

Termination and running time analysis

Lemma (Bound on RELABEL operations) The number of relabel operations is at most $2n - 1$ per vertex and at most $2n^2$ overall.

Proof: Any time during execution $h(u) \le 2n - 1$ for each vertex $u \in V$.

Lemma (Bound on saturating PUSHes) The number of saturating pushes is at most $2nm$.

Proof: Between two saturating pushes from u to v the height of v increases by at least 2.

Lemma (Bound on non-saturating PUSHes) The number of non-saturating pushes is at most $4n^2(n + m)$.

Proof: Estimate the change of the potential function $\Phi = \sum_{v \in V, e(v) > 0} h(v)$ during the three basic operations.

Theorem The number of basic operations for the `GENERIC-PUSH-RELABEL` algorithm is at most $O(n^2 m)$.

Corollary There is an implementation of the `GENERIC-PUSH-RELABEL` algorithm which runs in $O(n^2 m)$ on any flow network.

Admissable edges and admissable digraph

uv is an admissable edge if

- $c_f(uv) > 0$ and
- $h(u) = h(v) + 1$

Admissable digraph: $D_{f,h} = (V, E_{f,h})$, where $E_{f,h}$ is the set of admissable edges.

Lemma The admissable digraph is acyclic.

Observation If u is overflowing and uv is an admissable edge then PUSH(u, v) applies.
PUSH(u, v) does not create any new admissable edges, but it may cause uv to become inadmissable.

Observation If u is overflowing and there are no admissable edges leaving u, then RELABEL(u) applies. After RELABEL(u) there is at least one admissable edge leaving u and there are no admissable edges entering u.

Notation

D is given by (non-cyclic) neighbor lists:
$N(u)$ is the neighbor list of u
v is on $N(u)$ if uv or $vu \in E$

head$(N(u))$ points to the first vertex in $N(u)$
next-neighbor(v) points to the vertex following v in $N(u)$
next-neighbor(v) $=$NIL if v is the last vertex of $N(u)$
current(u) points to the neighbor u currently under consideration. Initially *current*(u) points to *head*$(N(u))$.

Recall – Basic operations

PUSH(u, v) applies if
u is overflowing and $uv \in E_{f,h}$. Then

$d_f(uv) :=$ min$\{e(u), c_f(uv)\}$
$f(uv) := f(uv) + d_f(uv)$
$f(vu) := -f(uv)$
$e(u) := e(u) - d_f(uv)$
$e(v) := e(v) + d_f(uv)$

RELABEL(u) applies if u is overflowing and $uv \notin E_{f,h}$ for all $v \in V$. Then

$h(u) := 1 +$ min$\{h(v) : uv \in E_f\}$

Discharging a vertex

DISCHARGE(u)

```
WHILE e(u) > 0
    DO v := current(u)
        IF v = NIL
            THEN RELABEL(u)
                current(u) := head(N(u))
        ELSEIF c_f(u, v) > 0 and h(u) = h(v) + 1
            THEN PUSH(u, v)
        ELSE current(u) := next-neighbor(v)
```

Lemma (Algorithm DISCHARGE is well-defined)
When DISCHARGE calls PUSH(u, v) then a push operation applies to uv.
When DISCHARGE calls RELABEL(u) then a relabel operation applies to u.

The RELABEL-TO-FRONT(D, c, s, t) algorithm

1 $f :=$ INITIALIZE-PREFLOW(D, c, s, t)
2 $L :=$ any order of $V \setminus \{s, t\}$
3 FOR each $u \in V \setminus \{s, t\}$
4   DO *current*(u) := *head*($N(u)$)
5 $u :=$ *head*(L)
6 WHILE $u \neq NIL$
7   DO *old-height* $:= h(u)$
8   DISCHARGE(u)
9   IF $h(u) >$ *old-height*
10   THEN move u to the front of the list L
11         $u :=$ *next*(u)

Theorem (Correctness)
The RELABEL-TO-FRONT algorithm is an implementation of the GENERIC-PUSH-RELABEL algorithm.

Proof. At each test in line 6 of the algorithm the list L is a topological sort of the vertices of the admissable digraph $D_{f,h}$ and no vertex in the list before u has excess flow.

Running time analysis

Theorem The running time of RELABEL-TO-FRONT on any flow network is $O(n^3)$

Proof

“Phase”: time between two relabel operations.

There are at most $O(n^2)$ relabel operations.
Hence there are at most $O(n^2)$ phases.

Each phase consists of $\leq n$ calls to DISCHARGE
Hence the total time of the WHILE loop excluding the work DISCHARGE does is $O(n^3)$.

Time spent during `DISCHARGE`

The $O(n^2)$ relabel operations can be done in $O(nm)$ (Homework)

Updating *current*(u) in line `8` of `DISCHARGE` occurs $O(\deg(u))$ times each time a vertex u is relabeled and $O(n \deg(u))$ times over for the vertex. All together the time is $O(nm)$ (Handshaking Lemma).

The overall number of saturating pushes is $O(nm)$

There is at most one non-saturating push per call to `DISCHARGE`
Hence the number of nonsaturating pushes is at most $O(n^3)$.