RECALL: Vertex coloring, chromatic number

A \( k \)-coloring of a graph \( G \) is a labeling \( f : V(G) \rightarrow S \), where \(|S| = k\). The labels are called colors; the vertices of one color form a color class.

A \( k \)-coloring is proper if adjacent vertices have different labels. A graph is \( k \)-colorable if it has a proper \( k \)-coloring.

The chromatic number is

\[
\chi(G) := \min\{k : G \text{ is } k\text{-colorable}\}.
\]

A graph \( G \) is \( k \)-chromatic if \( \chi(G) = k \). A proper \( k \)-coloring of a \( k \)-chromatic graph is an optimal coloring.

Examples. \( K_n, K_{n,m}, C_5 \), Petersen
RECALL: Lower bounds

Simple lower bounds

\[ \chi(G) \geq \omega(G) \]
\[ \chi(G) \geq \frac{n(G)}{\alpha(G')} \]

Examples for \( \chi(G') \neq \omega(G') \):

- odd cycles of length at least 5,
  \[ \chi(C_{2k+1}) = 3 > 2 = \omega(C_{2k+1}) \]

- complements of odd cycles of order at least 5,
  \[ \chi(\overline{C}_{2k+1}) = k + 1 > k = \omega(\overline{C}_{2k+1}) \]

- random graph \( G = G(n, \frac{1}{2}) \), almost surely
  \[ \chi(G') \approx \frac{n}{2 \log n} > 2 \log n \approx \omega(G) \]
RECALL: Upper bounds

**Proposition** \( \chi(G) \leq \Delta(G) + 1 \).

*Proof.* Algorithmic; Greedy coloring.

A graph \( G \) is *d-degenerate* if every subgraph of \( G \) has minimum degree at most \( d \).

**Claim.** \( G \) is \( d \)-degenerate iff there is an ordering of the vertices \( v_1, \ldots, v_n \), such that \( |N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leq d \)

**Proposition.** For a \( d \)-degenerate \( G \), \( \chi(G) \leq d + 1 \).

In particular, for every \( G \), \( \chi(G) \leq \max_{H \subseteq G} \delta(H) + 1 \).

*Proof.* Greedy coloring.

**Brooks’ Theorem.** (1941) Let \( G \) be a connected graph. Then \( \chi(G) = \Delta(G) + 1 \) iff \( G \) is a complete graph or an odd cycle.

*Proof.* Trickier, but still greedy coloring...
Equitable colorings

**Definition** A coloring of $G$ is **equitable** if it is proper and the size of any two color classes differ by at most one.

**Applications** Many...

**Conjecture** (Erdős, 1964) For each $r \geq \Delta(G)$, $G$ has an equitable $(r + 1)$-coloring.

**Remark** Strengthening of the greedy coloring upper bound $\chi(G) \leq \Delta(G) + 1$.

**Theorem** (Hajnal-Szemerédi, 1970) For each $r \geq \Delta(G)$, $G$ has an equitable $(r + 1)$-coloring.

**Proof.** Complicated, long (22 pages).

**Question** Is there a polynomial time algorithm which, given a graph $G$, finds an equitable $(\Delta(G) + 1)$-coloring of $G$?

The theorem is best possible: Let $G = \frac{n}{l}K_l$ for some integer $l|n$. There is no proper $(r + 1)$-coloring if $r = \Delta(G) - 1$.

Let $n$ be even. What does the HSz-Theorem says if $\Delta(G) \leq \frac{n}{2} - 1$?
There is an equitable $\frac{n}{2}$-coloring of $G$.

Or equivalently: In $\bar{G}$ there is a perfect matching.

**Special case 1** of HSzT: 
$\delta(H) \geq \frac{n}{2} \Rightarrow H$ has a perfect matching.

In fact, even more is true: **Dirac’s Theorem**: 
$\delta(H) \geq \frac{n}{2} \Rightarrow H$ has a Hamilton cycle!

**Special case 2.** (Corrádi-Hajnal Theorem) 
If $3|n$ and $\delta(H) \geq \frac{2n}{3}$, then $H$ has a $K_3$-factor (a family of triangles partitioning the vertex set).
Nearly equitable colorings

\[ |V(G)| = s(r + 1), \text{ where } r \geq \Delta(G). \]

An \((r + 1)\)-coloring \(f\) of \(G\) is nearly equitable if it is proper and all classes have the same size \(s\) except one small class \(V^- = V^-(f)\) with size \(s - 1\) and one large class \(V^+ = V^+(f)\) with size \(s + 1\).

Let \(U\) and \(W\) be two distinct color classes of \(f\). Vertex \(y \in U\) is movable to \(W\) if \(y\) has no neighbors in \(W\).

Auxiliary digraph \(H = H(G, f)\).

\(V(H) = \{U : U\ \text{is a color class of } f\}\).

\(UW \in E(H)\) if some vertex of \(U\) is movable to \(W\).

\(W\) is accessible if there is a \(WV^-\)-path in \(H\).

Remark \(V^-\) is accessible.

\(A = A(f)\) the family of accessible classes.
Accessible classes — how they can help

\[ A := \cup \mathcal{A}, \quad B := V(G) \setminus A \]

**Lemma 1** If \( G \) has a **nearly equitable** \((r + 1)\)-coloring \( f \) whose large class \( V^+ \in \mathcal{A} \), then \( G \) has an **equitable** \((r + 1)\)-coloring.

Hence assume \( V^+ \notin \mathcal{A} \)

\[ m := |\mathcal{A}| - 1 \]
\[ q := r - m \] the number of non-accessible classes

**Facts** \( |A| = (m + 1)s - 1 \)
\[ |B| = qs + 1 \]

\( y \in B \) cannot be moved to \( A \), so
\[ d_A(y) \geq m + 1 \], which implies \( d_B(y) \leq q - 1 \).

**Consequence** Kicking out any vertex \( y \) from \( B \) leaves us with a subgraph \( H = G[B \setminus \{y\}] \) having \( qs \) vertices and \( \Delta(H) \leq q - 1 \). An invitation for induction!
Terminal classes

$m \geq 1$, since otherwise $A = V^-$, so

$$rs + 1 \leq \sum_{y \in B} d_A(y) = |E(A, B)| = \sum_{x \in A} d_B(x) \leq r|V^-| = r(s - 1)$$

For $W \in \mathcal{A}$ let $\mathcal{A} \setminus \{W\} = S_W \cup T_W$, where $T_W = \{Z \in \mathcal{A} : \text{ every } ZV^-\text{-path goes through } W\}$.

$U \in \mathcal{A}$ is terminal if $T_U = \emptyset$, that is, there is a $ZV^-$-path avoiding $U$ for each $Z \in \mathcal{A} \setminus \{U\}$.

$U$ is non-terminal if $T_U \neq \emptyset$.

**Remark** $V^-$ is non-terminal

$S_{V^-} = \emptyset$, $T_{V^-} = \mathcal{A} \setminus \{V^-\}$.

Fix a non-terminal class $U \in \mathcal{A}$ and let $\mathcal{A}' := T_U$. 
Solo vertices

\[ t := |A'|, \quad A' := \cup A' \]

\[ x \in A' \Rightarrow x \text{ is not movable to any class in } A \setminus A' \setminus \{U\} \Rightarrow d_A(x) \geq m - t \]

zy is a solo edge and z and y are solo vertices if

- \( y \in B \)
- \( z \in W \in A' \)
- \( N_W(y) = \{z\} \)

Remark: y is movable to \( W \setminus \{z\} \).

\[ S_z := \{y \in B : \text{zy is a solo edge}\} \]
\[ S_y := \{z \in A' : \text{zy is a solo edge}\} \]

Claim \( y \in B \Rightarrow |S_y| \geq t - q + 1 + d_B(y). \)

Proof: \( t - |S_y| \leq |\{Z \in A : |Z \cap N(y)| \geq 2\}| \leq r - d_B(y) - (m + 1) \)
Lemmas

Lemma 2 If there exists \( W \in \mathcal{A}' \) such that no solo vertex in \( W \) is movable to a class in \( \mathcal{A} \setminus \{W\} \) then \( q + 1 \leq t \). Furthermore, every vertex of \( B \) is solo.

Remark: For any solo vertex \( z \in W \) there is a \( y \in B \) which we could move to \( W \) — should we be able to get rid of \( z \) by moving it further. Lemma 2 discusses the “bad case”, when this is not possible for any solo vertex \( z \).

Lemma 3 Let \( W \in \mathcal{A}' \). Then \( \exists \) a solo vertex \( z \in W \) such that either \( z \) is movable to a class in \( \mathcal{A} \setminus \{W\} \) or \( S_z \) is not a clique.
Proof of Lemma 2

**Lemma 2** If there exists $W \in \mathcal{A}'$ such that no solo vertex in $W$ is movable to a class in $\mathcal{A} \setminus \{W\}$ then $q + 1 \leq t$. Furthermore, every vertex of $B$ is solo.

**Proof.** Doublecount $|E(W, B)|$.

$S \subseteq W$ set of solo vertices in $W$, $D = W \setminus S$.

No $z \in S$ is movable to $\mathcal{A} \setminus \{W\} \Rightarrow d_{A}(z) \geq m$

$\Rightarrow d_{B}(z) \leq q$

No $z \in D$ is movable to $\mathcal{A} \setminus \mathcal{A}' \setminus \{U\} \Rightarrow d_{B}(z) \leq q + t$

$|N_{B}(S)| + 2(|B| - |N_{B}(S)|) \leq |E(W, B)|$

$\leq q|S| + (t + q)|D|$

$2(qs + 1) - q|S| \leq qs + t|D|$

$q + \frac{2}{|D|} \leq t$

By Claim $|S^{y}| \geq t - q + 1 + d_{B}(y) \geq 2$. \qed
Proof of Lemma 3

**Lemma 3** Let $W \in A'$. Then $\exists$ a solo vertex $z \in W$ such that either $z$ is movable to a class in $A \setminus \{W\}$ or $S_z$ is *not* a clique.

**Proof.** Suppose the statement is false.

- Lemma 2 $\Rightarrow \forall y \in B$ is solo and $t - q \geq 1$.
- $S_z$ is a clique $\Rightarrow \forall y \in S_z$, $d_B(y) + 1 \geq |S_z|$.

$$
\mu(xy) = \begin{cases} 
\frac{q}{|S_x|} & \text{if } xy \text{ is solo} \\
0 & \text{otherwise}
\end{cases}
$$

Doublecount $\mu(A', B) = \sum_{x \in A'} \sum_{y \in B} \mu(xy)$.

$$
\sum_{x \in A'} \sum_{y \in B} \mu(xy) = \sum_{\text{solo } z \in A'} |S_z| \cdot \frac{q}{|S_z|} \leq qst.
$$

$$
\sum_{y \in B} \sum_{x \in A'} \mu(xy) = \sum_{y \in B} \sum_{z \in S_y} \frac{q}{|S_z|} \geq \sum_{y \in B} |S^y| \frac{q}{c_y}
$$

$$
\geq \sum_{y \in B} (t - q + c_y) \frac{q}{c_y}
$$

$$
\geq \sum_{y \in B} t = t|B| = t(qs + 1)
$$

* $c_y = \max\{|S_z| : z \in S^y\} \leq d_B(y) + 1 \leq q$
Proof of the Hajnal-Szemerédi Theorem

**Theorem** (Hajnal-Szemerédi, 1970) For each \( r \geq \Delta(G) \), \( G \) has an equitable \((r + 1)\)-coloring.

**Proof:**
WLOG \( n = s(r + 1) \). Let \( G \) be a counterexample on \( n \) vertices with the smallest number of edges.

**Consequence:** For arbitrary edge \( e = xy \in E(G) \),

- there is an equitable \((r + 1)\)-coloring \( f_0 \) of \( G - e \)
- \( x \) and \( y \) must be in the same color class \( V \) of \( f_0 \).

\[ d(x) \leq r \implies \exists \text{ class } W \neq V, x \text{ is movable to } W. \]
\[ \implies \exists \text{ a nearly equitable } (r + 1)\text{-coloring of } G. \]

Let \( f \) be a nearly equitable \((r + 1)\)-coloring of \( G \) such that the number \( q = q(f) \) of nonaccessible classes is minimal.

\[ \text{L1 } \implies V^+ \not\in A = A(f) \]
Proof of Hajnal-Szemerédi Theorem – cont’d

Fix non-terminal class \( U \in \mathcal{A} \) with \( I_U =: \mathcal{A}' \) minimal. Recall: \( \mathcal{A}' \neq \emptyset \)
Minimality \( \Rightarrow \) every class in \( \mathcal{A}' \) is terminal

L3 \( \Rightarrow \exists \) class \( W \in \mathcal{A}', \) a solo vertex \( z \in W \) and a vertex \( y_1 \in S_z \) such that

- either \( z \) is movable to a class \( X \in \mathcal{A} \setminus \{W\} \)
- or \( z \) is not movable in \( \mathcal{A} \) and there exists another vertex \( y_2 \in S_z \) which is not incident to \( y_1 \).

Recall \( A = \bigcup \mathcal{A}, \quad B = V(G) \setminus A \)

\( A^+ := A \cup \{y_1\}, \quad B^- := B \setminus \{y_1\} \)

Recall: there exists an equitable \( q \)-coloring \( g \) of \( G[B^-] \)

\( (d_B(y) \leq q - 1, \forall y \in B \Rightarrow \) induction applies)
Proof of Hajnal-Szemerédi Theorem – Cases

Case 1. $z$ is movable to $X \in \mathcal{A}$.

$W$ is terminal $\Rightarrow \exists X V^-$-path in $H$ avoiding $W$.
Move vertices along this path, $z$ to $X$, $y_1$ to $W \setminus \{z\}$
This creates an equitable $m+1$-coloring $\varphi'$ of $G[A^+]$.
Then $\varphi' \cup g$ is an equitable $(r+1)$-coloring of $G$, a contradiction.

Case 2. $z$ is not movable to any class in $\mathcal{A}$.

Then $d_{B^-}(z) \leq q - 1$ and $z$ can be moved into a color class $Y \subseteq B^-$ of $g$. This defines an equitable $q$-coloring $g'$ of $G$ on $B^* = B^- \cup \{z\}$
Move $y_1$ to $W \setminus \{z\}$ to obtain an equitable $(m+1)$-coloring $\psi$ of $G$ on $A^* = V(G) \setminus B^*$.
$\psi' := \psi \cup g'$ is a nearly equitable $r+1$-coloring of $G$.

$A^* \subseteq A(\psi')$ and the class of $y_2$ is now accessible!
(since $y_2$ is movable to $W^* = W \cup \{y_1\} \setminus \{z\}$)
Thus $q(\psi') < q(f)$, a contradiction. $\square$
Fast equitable coloring

**Algorithm** EquiColor\((G, r)\) (Kostochka-Kierstead, 2006)

**Input.** graph \(G\), integer \(r \geq \Delta(G)\)

**Output.** equitable \((r + 1)\)-coloring of \(G\)

\(V(G) = \{v_1, \ldots, v_n\}, n = s(r + 1) - p\)

For \(i, 0 \leq i < n\), let \(G_i \subseteq G\), \(V(G_i) = V(G)\),
\(E(G_i) = \{xy : x \text{ or } y = v_j, j \leq i\}\)

IF \(p \neq 0\) THEN

**output** EquiColor\((G + K_p, r)|_{V(G)}\)

ELSE

\(i := 0, f_0 := \text{arbitrary equitable } r\)-coloring of \(G_0\)

WHILE \(i < n - 1\) DO \(i := i + 1\)

IF \(v_i\) has no \(G\)-neighbors in its \(f_{i-1}\)-color class THEN

\(f_i := f_{i-1}\)

ELSE Define \(f'_i\) from \(f_{i-1}\) by moving \(v_i\) to

a \(f_{i-1}\)-color class that has no \(G\)-neighbors of \(v_i\)

\(f_i := \text{Equitizer}(G_i, f'_{i-1})\)

**output** \(f_{n-1}\).

**Theorem.** EquiColor\((G, r)\) outputs a nearly equitable \((r + 1)\)-coloring of \(G\) in time \(O(n^5)\).
Algorithm \textbf{Equitizer}(G, f)

\textbf{Input} graph \( G \) on \( n := s(r + 1) \) vertices, \( \Delta(G') \leq r \); nearly equitable \((r + 1)\)-coloring \( f \) of \( G \)

\textbf{Output} equitable \((r + 1)\)-coloring \( f' \) of \( G \)

Construct auxiliary digraph \( H = H(f) \)

\textbf{IF} \( V^+ \in \mathcal{A} \) \textbf{THEN}

\hspace{1cm} \( f' := \) recoloring of \( f \) by moving vertices along a \( V^+V^- \)-path in \( H \)

\textbf{ELSE}

\hspace{1cm} Construct \( \mathcal{A}, \mathcal{A}', B, W, z, y_1 \) as in the proof

\hspace{1cm} \( g := \text{Equitizer}(G[B^-], f|_{B^-}) \)

\hspace{1cm} \textbf{IF} \( z \) is movable to an \( X \in \mathcal{A} \) (i.e., Case 1) \textbf{THEN}

\hspace{2cm} construct \( \varphi' \); \( f' := \varphi' \cup g \)

\hspace{1cm} \textbf{ELSE} (i.e., Case 2)

\hspace{2cm} find \( y_2 \in S_z \) such that \( y_1y_2 \notin E \)

\hspace{2cm} construct \( g' \) on \( B^* \) and \( \phi \) on \( A^* \)

\hspace{2cm} construct nearly equitable \( \psi' := \psi \cup g' \)

\hspace{2cm} \( f' := \text{Equitizer}(G, \psi') \) [choose \( \mathcal{A}'(\psi') \subseteq B(f) \)]

\hspace{1cm} \textbf{output} \( f' \)
Running time

**Theorem.** There exists a constant $c$ such that algorithm $\text{Equitizer}(G, f)$ outputs an equitable $(r + 1)$-coloring of $G$ in time $c(q + 1)n^3$, where $q = q(f)$ is the number of non-accessible classes of $f$.

**Corollary.** Algorithm $\text{EquiColor}(G, r)$ constructs an equitable $(r + 1)$-coloring of graph $G$ in time $O(n^5)$. 
Running time analysis

**Proof of Theorem.** Termination of Equitizer is clear.

\[ R(G, f) := \text{runtime of Equitizer}(G, f) \]

\[ R(n, q) := \max \{ R(G, f) : |V(G)| = n, q(f) \leq q \} \]

Let \( c \) be a constant such that all of the lines of Equitizer not calling itself recursively (i.e. recolorings, searches, constructions, case-determinations) can be performed in time \( \frac{c}{2} n^3 \).

Induction on \( q = q(f) \):

\( q = 0 \Rightarrow V^+ \in \mathcal{A} \) and \( R(n, 0) \leq cn^3 \).

Assume \( q > 0 \)

If Case 1 happens:

\[ R(n, q) \leq \frac{c}{2} n^3 + R(|B^-|, q - 1) \leq cn^3 + cqn^3 \leq c(q + 1)n^3 \]
Running time analysis — cont’d

If Case 2 happens: \( q(\psi') < q \) and
\[
R(n, q) \leq \frac{c}{2}n^3 + R(|B^-|, q - 1) + R(G, \psi')
\]

Case 2 \( \Rightarrow \) L2 \( \Rightarrow \) \( q + 1 \leq t \) \( \Rightarrow \) \( |B^-| \leq \frac{n}{2} \)
\[
\Rightarrow R(|B^-|, q - 1) \leq cq \left(\frac{n}{2}\right)^3
\]

Using \( R(G, \psi') \leq cqn^3 \) would not suffice.

We go one deeper into the algorithm.

If, after Case 2, Case 1 happens:
Because \( |B^- (\psi')| \leq |B^- (f)| \), we have
\[
R(G, \psi') \leq \frac{c}{2}n^3 + R(|B^- (\psi')|, q(\psi'))
\leq \frac{c}{2}n^3 + cq \left(\frac{n}{2}\right)^3
\]

If, after Case 2, Case 2 happens again:
\[
\mathcal{A'}(\psi') \subseteq B(f) \Rightarrow q(\psi') + t(\psi') \leq q(f)
\]
L2 \( \Rightarrow \) \( q(\psi') + 1 \leq t(\psi') \) \( \Rightarrow \) \( q(\psi') \leq \frac{q(f) - 1}{2} \).

\[
R(G, \psi') \leq R(n, q/2) \leq c\frac{q+1}{2}n^3
\]