

When is a graph planar?_____

Theorem(Euler, 1758) If a plane multigraph G with k components has n vertices, e edges, and f faces, then

$$n - e + f = 1 + k.$$

Corollary If G is a simple, planar graph with $n(G) \geq 3$, then $e(G) \leq 3n(G) - 6$.

If also G is triangle-free, then $e(G) \leq 2n(G) - 4$.

Corollary K_5 and $K_{3,3}$ are non-planar.

The **subdivision of edge** $e = xy$ is the replacement of e with a new vertex z and two new edges xz and zy . The graph H' is a **subdivision of H** , if one can obtain H' from H by a series of edge subdivisions. Vertices of H' with degree at least three are called **branch vertices**.

Theorem(Kuratowski, 1930) A graph G is planar **iff** G does not contain a subdivision of K_5 or $K_{3,3}$.

Kuratowski's Theorem_____

Theorem(Kuratowski, 1930) A graph G is planar **iff** G does not contain a subdivision of K_5 or $K_{3,3}$.

Proof.

A **Kuratowski subgraph** of G is a subgraph of G that is a subdivision of K_5 or $K_{3,3}$. A **minimal nonplanar graph** is a nonplanar graph such that every proper subgraph is planar.

A **counterexample** to Kuratowski's Theorem constitutes a **nonplanar** graph that does **not** contain any Kuratowski subgraph.

Kuratowski's Theorem follows from the following Main Lemma and Theorem.

The spine of the proof_____

Main Lemma. If G is a graph with fewest edges among counterexamples, then G is 3-connected.

Lemma 1. Every minimal nonplanar graph is 2-connected.

Lemma 2. Let $S = \{x, y\}$ be a separating set of G . If G is a nonplanar graph, then adding the edge xy to some S -lobe of G yields a nonplanar graph.

Main Theorem.(Tutte, 1960) If G is a 3-connected graph with no Kuratowski subgraph, then G has a convex embedding in the plane with no three vertices on a line.

A **convex embedding** of a graph is a planar embedding in which each face boundary is a convex polygon.

Lemma 3. If G is a 3-connected graph with $n(G) \geq 5$, then there is an edge $e \in E(G)$ such that $G \cdot e$ is 3-connected.

Notation: $G \cdot e$ denotes the graph obtained from G after the **contraction** of edge e .

Lemma 4. G has no Kuratowski subgraph $\Rightarrow G \cdot e$ has no Kuratowski subgraph.

Proof of Tutte's Theorem

Main Theorem. (Tutte, 1960) If G is a 3-connected graph with no Kuratowski subgraph, then G has a convex embedding in the plane with no three vertices on a line.

Proof. Induction on $n(G)$.

Base case: G is 3-connected, $n(G) = 4 \Rightarrow K_4$.

Let $e \in E$ s.t. $H = G \cdot e$ is 3-connected. (Lemma 3)

Then H has no Kuratowski subgraph. (Lemma 4)

Induction $\Rightarrow H$ has a convex embedding in the plane with no three vertices on a line.

Let $z \in V(H)$ be the contracted e .

$H - z$ is 2-connected \Rightarrow boundary of the face containing z after the deletion of the edges incident to z is a cycle C .

Let x_0, \dots, x_{k-1} be the neighbors of x on C in cyclic order. Note that $|N(x)| \geq 3$ and hence $k \geq 2$.

Denote by $\langle x_i, x_{i+1} \rangle$ the portion of C from x_i to x_{i+1} (including endpoints; indices taken modulo k).

Let $N_x = N(x) \setminus \{y\}$ and $N_y = N(y) \setminus \{x\}$.

Case 1. $|N_x \cap N_y| \geq 3$.

Let $u, v, w \in N_x \cap N_y$. Then x, y, u, v, w are the branch vertices of a K_5 -subdivision in G .

Case 2. $|N_x \cap N_y| \leq 2$.

Since $|N_x \cup N_y| \geq 3$, there is w.l.o.g. a vertex $u \in N_y \setminus N_x$. Let i be such that u is on $\langle x_i, x_{i+1} \rangle$.

Case 2a. N_y is contained in $\langle x_i, x_{i+1} \rangle$.

Then there is an appropriate embedding of G : Placing x in place of z and y sufficiently close to x maintains convexity. (No three vertices are collinear; $|N(x)|, |N(y)| \geq 3$.)

Case 2b. For every i there is a vertex in N_y that is not in $\langle x_i, x_{i+1} \rangle$.

Then there must be a $v \in N_y$ that is not on $\langle x_i, x_{i+1} \rangle$ and x, y, x_i, x_{i+1}, u, v are the branch vertices of a $K_{3,3}$ -subdivision in G .

Proof of the Lemmas

Lemma 3. G is 3-connected, $n(G) \geq 5 \Rightarrow$ there is an edge $e \in E(G)$ such that $G \cdot e$ is 3-connected.

Proof. Suppose G is 3-connected and for every $e \in E$, $G \cdot e$ is NOT 3-connected.

For edge $e = xy$, the vertex z is the mate of xy if $\{x, y, z\}$ is a cut in G .

Choose $e = xy$ and their mate z such that $G - \{x, y, z\}$ has a component H , whose order is as large as possible.

Let H' be another component of $G - \{x, y, z\}$ and let $u \in V(H')$ be the neighbor of z (There IS one!). Let v be the mate of uz .

$V(H) \cup \{x, y\} \setminus \{v\}$ is connected in $G - \{z, u, v\}$ contradicting the maximality of H . \square

Lemma 4. G has no Kuratowski subgraph $\Rightarrow G \cdot e$ has no Kuratowski subgraph.

Proof. Suppose $G \cdot e$ contains a Kuratowski subgraph H . Then

- $z \in V(H)$
- z is a branchvertex of H
- $|N_H(z)| = 4$ and $|N_H(z) \cap N_G(x)|, |N_H(z) \cap N_G(y)| \geq 2$

Then H is the subdivision of $K_5 \Rightarrow G$ contains a subdivision of $K_{3,3}$, a contradiction. \square

Minors

K_7 is a toroidal graph (it is embeddable on the torus), K_8 is not. What else is not? For the torus there is NO equivalent version of Kuratowski's characterization with a finite number of forbidden subdivisions. Any such characterization must lead to an infinite list.

A weaker concept: Minors.

Graph G is called a **minor of graph H** if G can be obtained from H by a series of edge deletions and edge contractions. Graph H is also called a **G -minor**

Example: K_5 is a minor of the Petersen graph P , but P does not contain a K_5 -subdivision.

The Graph Minor Theorem_____

Theorem. (Robertson and Seymour, 1985-2005) In any infinite list of graphs, some graph is a minor of another.

Proof: more than 500 pages in 20 papers.

Corollary For any graph property that is closed under taking minors, there exists **finitely many** minimal **forbidden** minors.

Homework. Wagner's Theorem. Every nonplanar graph contains either a K_5 or $K_{3,3}$ -minor.

For embeddability on the **projective plane**, it is known that there are **35** minimal forbidden minors. For embeddability on the **torus**, we don't know the exact number of minimal forbidden minors; there are **more than 800 known**.