RECAP — Matchings

A matching is a set of (non-loop) edges with no shared endpoints. The vertices incident to an edge of a matching $M$ are saturated by $M$, the others are unsaturated. A perfect matching of $G$ is matching which saturates all the vertices.

Examples. $K_{n,m}$, $K_n$, Petersen graph, $Q_k$; graphs without perfect matching

The size of the largest matching in $G$ is denoted by $\alpha'(G)$.

A vertex cover of $G$ is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge. The size of the smallest vertex cover in $G$ is $\beta(G)$.

Claim. For every graph $G$, $\beta(G) \geq \alpha'(G)$. 
Suppose we knew that in some graph $G$ with 1121 edges on 200 vertices, a particular set of 87 edges is (one of) the largest matching one could find. How could we convince somebody about this?

Once the particular 87 edges are shown, it is easy to check that they are a matching, indeed.

But why isn’t there a matching of size 88? Verifying that none of the $\binom{1121}{88}$ edgesets of size 88 forms a matching could take some time...

If we happen to be so lucky, that we are able to exhibit a vertex cover of size 87, we are saved. It is then reasonable to check that all 1121 edges are covered by the particular set of 87 vertices.

Exhibiting a vertex cover of a certain size proves that no larger matching can be found.
Certificate for bipartite graphs — Take 1

1. **Correctness** of the certificate:
A vertex cover $Q \subseteq V(G)$ is a certificate proving that no matching of $G$ has size larger than $|Q|$.
That is: $\beta(G') \geq \alpha'(G')$, valid for every graph.

2. **Existence** of optimal certificate for bipartite graphs:
**Theorem.** (König (1931), Egerváry (1931))
If $G$ is bipartite then $\beta(G') = \alpha'(G)$.

König’s Theorem $\Rightarrow$ For bipartite graphs there always exists a vertex cover proving that a particular matching of maximum size is really maximum.

**Remark.** This is **NOT** the case for general graphs: $C_5$. 


Certificate for **bipartite** graphs — Take 2

Let $G$ be a bipartite graph with partite sets $X$ and $Y$.

1. **Correctness** of the certificate:
   A subset $S \subseteq X$ is a certificate proving that the largest matching in $G$ has size at most $|X| - |S| + |N(S)|$.

2. **Existence** of optimal certificate:
   **Theorem** (Marriage Theorem; Hall, 1935) There is a matching in $G$ saturating $X$ iff $|N(S)| \geq |S|$ for every $S \subseteq X$.
   **Corollary** $\alpha'(G) = |X|$ or there exists a subset $S \subseteq X$, such that $\alpha'(G) = |X| - |S| + |N(S)|$.
   **Proof.** Homework.

**Problem:** Certificate makes sense for bipartite graphs only.

**Goal:** Find a certificate for general graphs.
Matchings in general graphs

An odd component is a connected component with an odd number of vertices. Denote by \( o(G) \) the number of odd components of a graph \( G \).

**Theorem.** (Tutte, 1947) A graph \( G \) has a perfect matching iff \( o(G - S) \leq |S| \) for every subset \( S \subseteq V(G) \).

**Proof.**

\[ \Rightarrow \text{ Easy.} \]

\[ \Leftarrow (\text{Lovász, 1975}) \text{ Consider a counterexample } G \text{ with the maximum number of edges.} \]

**Claim.** \( G + xy \) has a perfect matching for any \( xy \notin E(G) \).
Proof of Tutte’s Theorem — Continued

Define \( U := \{ v \in V(G) : d_G(v) = n(G) - 1 \} \)

**Case 1.** \( G - U \) consists of disjoint cliques.

*Proof:* Straightforward to construct a perfect matching of \( G \).

**Case 2.** \( G - U \) is not the disjoint union of cliques.

*Proof:* Derive the existence of the following subgraph.

Obtain contradiction by constructing a perfect matching \( M \) of \( G \) using perfect matchings \( M_1 \) and \( M_2 \) of \( G + xz \) and \( G + yw \), respectively.
Corollaries

**Corollary.** (Berge, 1958) For a subset $S \subseteq V(G)$ let $d(S) = o(G - S) - |S|$. Then

$$2\alpha'(G') = \min\{n - d(S) : S \subseteq V(G')\}.$$ 

*Proof.*  $(\leq)$ Easy.

$(\geq)$ Apply Tutte’s Theorem to $G \vee K_d$.

**Corollary.** (Petersen, 1891) Every 3-regular graph with no cut-edge has a perfect matching.

*Proof.* Check Tutte’s condition. Let $S \subseteq V(G')$. Double-count the number of edges between an $S$ and the odd components of $G - S$. Observe that between any odd component and $S$ there are at least three edges.