Graphs & Algorithms: Advanced Topics

Basic concepts, definitions and facts

Things

Prerequisite: basic graph theory and graph algorithms. In particular the material of the course Graphs and Algorithms (Spring 2007)

In graph theoretic notation we mostly follow the book "Introduction to Graph Theory" by Doug West.

Graphs – Definition

A graph G is a pair consisting of

- a vertex set $V(G)$, and
- an edge set $E(G) \subseteq \binom{V(G)}{2}$.

If there is no confusion about the underlying graph we often just write $V = V(G)$ and $E = E(G)$.

x and y are the endpoints of edge $e = \{x, y\}$.
They are called adjacent or neighbors.
e is called incident with x and y.

A loop is an edge whose endpoints are equal.
Multiple edges have the same set of endpoints. In the definition of a “graph” we don’t allow loops and multiple edges. To emphasize this, we often say “simple graph”. When we do want to allow multiple edges or loops, we say multigraph.

Remarks A multigraph might have no multiple edges or loops. Every (simple) graph is a multigraph, but not every multigraph is a (simple) graph.

Special graphs

K_n is the complete graph on n vertices.

$K_{n,m}$ is the complete bipartite graph with partite sets of sizes n and m.

P_n is the path on n vertices

C_n is the cycle on n vertices

Further definitions and notation

The degree of vertex v is the number of edges incident with v. Loops are counted twice.

A set of pairwise adjacent vertices in a graph is called a clique. A set of pairwise non-adjacent vertices in a graph is called an independent set.

A graph G is bipartite if $V(G)$ is the union of two (possibly empty) independent sets of G. These two sets are called the partite sets of G.

The complement $\overline{G}$ of a graph G is a graph with

- vertex set $V(\overline{G}) = V(G)$ and
- edge set $E(\overline{G}) = \binom{V}{2} \setminus E(G)$.

H is a subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. We write $H \subseteq G$. We also say G contains H and write $G \supseteq H$.

For a subset $S \subseteq V(G)$ define $G[S]$, the induced subgraph of G on S: $V(G[S]) = S$ and $E(G[S]) = \{e \in E(G) :$ both endpoints are in $S\}$.

Leaves, trees, forests...

A graph with no cycle is acyclic. An acyclic graph is called a forest.

A connected acyclic graph is a tree.

A leaf (or pendant vertex) is a vertex of degree 1.

A spanning subgraph of G is a subgraph with vertex set $V(G)$.

A spanning tree is a spanning subgraph which is a tree.

Examples. Paths, stars

Properties of trees

Lemma. T is a tree, $n(T) \geq 2 \Rightarrow T$ contains at least two leaves.
Deleting a leaf from a tree produces a tree.

Theorem (Characterization of trees) For an n-vertex graph G, the following are equivalent

1. G is connected and has no cycles.
2. G is connected and has $n-1$ edges.
3. G has $n-1$ edges and no cycles.
4. For each $u, v \in V(G)$, G has exactly one u, v-path.

Corollary.

(i) Every edge of a tree is a cut-edge.

(ii) Adding one edge to a tree forms exactly one cycle.

(iii) Every connected graph contains a spanning tree.

Walks, trails, paths, and cycles

A walk is an alternating list $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ of vertices and edges such that for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i.

A trail is a walk with no repeated edge.

A path is a walk with no repeated vertex.

A u, v-walk, u, v-trail, u, v-path is a walk, trail, path, respectively, with first vertex u and last vertex v.

If $u = v$ then the u, v-walk and u, v-trail is closed. A closed trail (without specifying the first vertex) is a circuit. A circuit with no repeated vertex is called a cycle.

The length of a walk trail, path or cycle is its number of edges.

Connectivity

G is connected, if there is a u, v-path for every pair $u, v \in V(G)$ of vertices.
Otherwise G is disconnected.

Vertex u is connected to vertex v in G if there is a u, v-path. The connection relation on $V(G)$ consists of the ordered pairs (u, v) such that u is connected to v.

Claim. The connection relation is an equivalence relation.

Lemma. Every u, v-walk contains a u, v-path.

The connected components of G are its maximal connected subgraphs (i.e. the equivalence classes of the connection relation).

An isolated vertex is a vertex of degree 0. It is a connected component on its own, called trivial connected component.

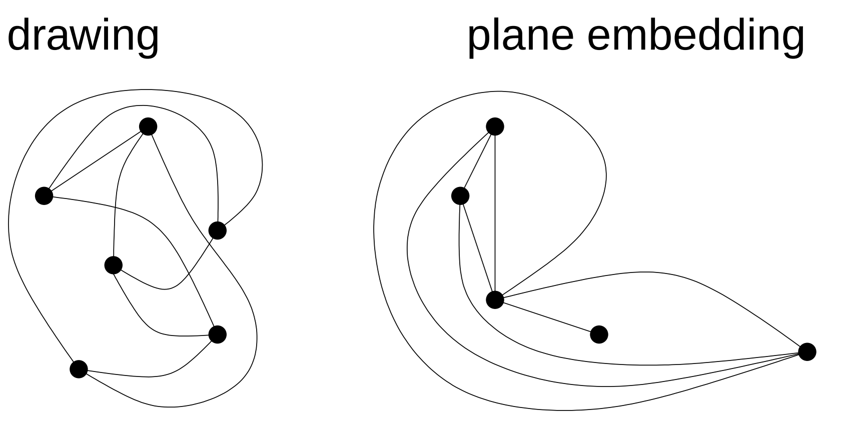
Planar graphs — a recap

A drawing of a multigraph G is a function f defined on $V(G) \cup E(G)$ that assigns

- a point $f(v) \in \mathbb{R}^2$ to each vertex v and
- an $f(u), f(v)$-curve to each edge uv,

such that the images of vertices are distinct. A point in $f(e) \cap f(e')$ that is not a common endpoint is a crossing.

A graph is planar if it has a drawing without crossings. Such a drawing is a planar embedding of G. A planar (multi)graph *together* with a particular planar embedding is called a plane (multi)graph.



Jordan curves

A curve is a subset of $I\!R^2$ of the form

$$\alpha = \{\gamma(x) : x \in [0,1]\} ,$$

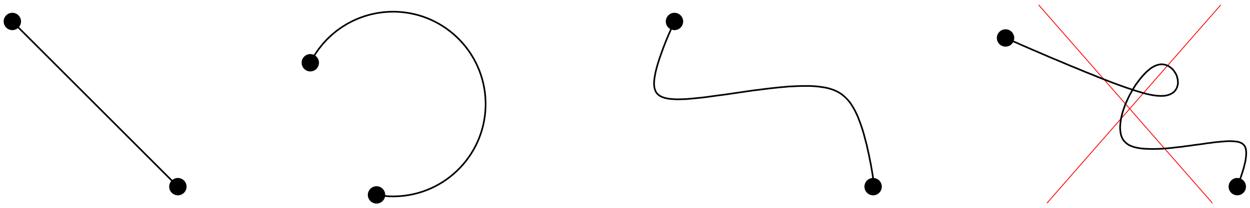
where $\gamma : [0,1] \to I\!R^2$ is a continuous mapping from the closed interval $[0,1]$ to the plane. $\gamma(0)$ and $\gamma(1)$ are called the *endpoints* of curve α.

A curve is closed if its first and last points are the same. A curve is simple if it has no repeated points except possibly first = last. A closed simple curve is called a Jordan-curve.

Examples: Line segments between $p, q \in I\!R^2$

$$x \mapsto xp + (1-x)q ,$$

circular arcs, Bezier-curves without self-intersection, etc...

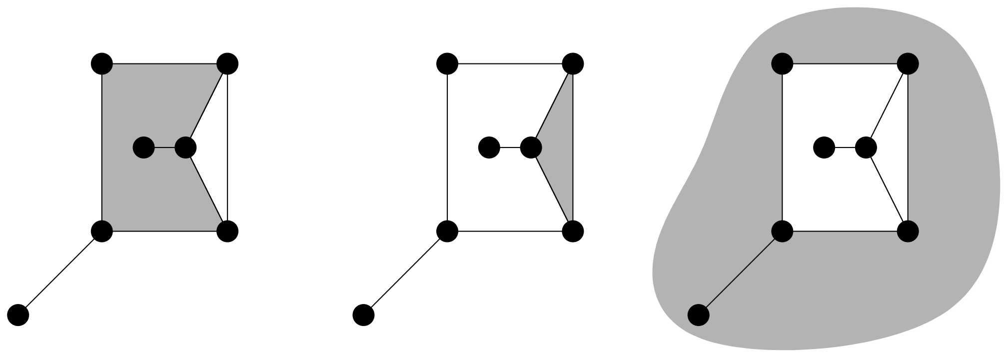


Regions and faces

An open set in the plane is a set $U \subseteq R^2$ such that for every $p \in U$, all points within some small distance belong to U. A region is an open set U that contains a u, v-curve for every pair $u, v \in U$.

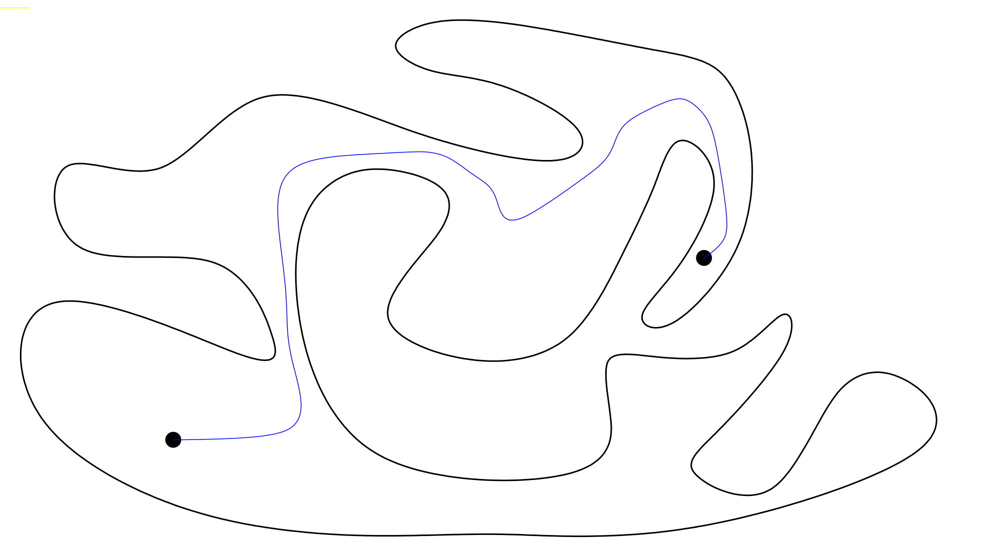
The faces of a plane multigraph are the maximal regions of the plane that contain no points used in the embedding.

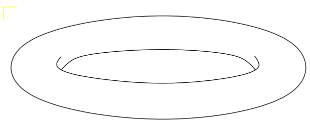
A finite plane multigraph G has one unbounded face (also called outer face).



The unconscious ingredient.

Jordan Curve Theorem. A simple closed curve C partitions the plane into exactly two faces, denoted by $\mathtt{Int}(C)$ and $\mathtt{Ext}(C)$, each having C as boundary.



Not true on the torus! 

13-a

Euler's Formula

Theorem. (Euler, 1758) If a plane multigraph G with k components has n vertices, e edges, and f faces, then

$$n - e + f = 1 + k.$$

Applications If G is a simple, planar graph with $n(G) \geq 3$, then

$$e(G) \leq 3n(G) - 6.$$

If G is a simple plane graph with $n(G) \geq 3$ vertices then

$$f(G) \leq 2n(G) - 4.$$