Graphs & Algorithms II  Exercise Set 5  HS07

URL: http://www.ti.inf.ethz.ch/ew/courses/GA07/

In this series of exercises you are asked to show how a tree decomposition of small width can be constructed for a given graph.

For a graph $G = (V, E)$ a set $U \subseteq V$ is $k$-linked if $|U| \geq k$ and there are no three subsets $A, B \subseteq U$ and $C \subseteq V$ such that

- $|C| < |A| = |B| \leq k$ and
- there is no path from $A \setminus C$ to $B \setminus C$ in $G \setminus C$.

Homework 5

Prove that if $G$ contains a $(k+1)$-linked set of size at least $3k$, $k \in \mathbb{N}$, then $G$ has treewidth at least $k$.

Exercise 14

Given a graph $G = (V, E)$, a set $U \subseteq V$ of $k$ vertices, $k \in \mathbb{N}$, and $\ell \leq k$. Show how to determine in $f(k)p(|V|)$ time whether $U$ is $\ell$-linked, where $p(\cdot)$ is a polynomial.

Moreover, if $U$ is not $\ell$-linked, the algorithm should return a certificate, that is, three sets $A, B \subseteq U$ and $C \subseteq V$ such that $|C| < |A| = |B| \leq \ell$ and there is no path from $A \setminus C$ to $B \setminus C$ in $G \setminus C$.

Exercise 15

Describe an algorithm that for a given graph $G = (V, E)$ and $k \in \mathbb{N}$ in $f(k)p(|V|)$ time

- either returns a tree decomposition of width at most $4k$ for $G$
- or returns a $(k+1)$-linked set $U \subseteq V$ of size at least $3k$,

where $p(\cdot)$ is a polynomial.

Hint. Construct a tree decomposition $T = ([X_i | i \in I], T = (I, F))$ incrementally, subject to the following invariants.

1. $T$ is a tree decomposition of width at most $4k$ for $G[U]$, where $U = \bigcup_{i \in I} X_i \subseteq V$.
2. Each component $C$ in $G \setminus U$ has at most $3k$ neighbors in $U$, and there is a single bag $X_i$, for some $i \in I$, that contains all of them.

Homework due: 31.10.2007, 11:00 AM.