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Graphs & Algorithms II Exercise Set 5 HS07

URL: http://www.ti.inf.ethz.ch/ew/courses/GA07/

In this series of exercises you are asked to show how a tree decomposition of small width can be constructed for a given graph.

For a graph $G = (V, E)$ a set $U \subseteq V$ is k-*linked* if $|U| \geq k$ and there are no three subsets $A, B \subseteq U$ and $C \subset V$ such that

- $|C| < |A| = |B| \leq k$ and
- there is no path from $A \setminus C$ to $B \setminus C$ in $G \setminus C$.

Homework 5

Prove that if G contains a $(k+1)$-linked set of size at least $3k$, $k \in \mathbb{N}$, then G has treewidth at least k.

Exercise 14

Given a graph $G = (V, E)$, a set $U \subseteq V$ of k vertices, $k \in \mathbb{N}$, and $\ell \leq k$. Show how to determine in $f(k)p(|V|)$ time whether U is ℓ-linked, where $p(\cdot)$ is a polynomial.

Moreover, if U is not ℓ-linked, the algorithm should return a certificate, that is, three sets $A, B \subseteq U$ and $C \subset V$ such that $|C| < |A| = |B| \leq \ell$ and there is no path from $A \setminus C$ to $B \setminus C$ in $G \setminus C$.

Exercise 15

Describe an algorithm that for a given graph $G = (V, E)$ and $k \in \mathbb{N}$ in $f(k)p(|V|)$ time

- either returns a tree decomposition of width at most $4k$ for G
- or returns a $(k+1)$-linked set $U \subseteq V$ of size at least $3k$,

where $p(\cdot)$ is a polynomial.

Hint. Construct a tree decomposition $\mathcal{T} = (\{X_i \mid i \in I\}, T = (I, F))$ incrementally, subject to the following invariants.

1. $\mathcal{T}$ is a tree decomposition of width at most $4k$ for $G[U]$, where $U = \bigcup_{i \in I} X_i \subseteq V$.
2. Each component C in $G \setminus U$ has at most $3k$ neighbors in U, and there is a single bag X_i, for some $i \in I$, that contains all of them.

Homework due: 31.10.2007, 11:00AM.