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Graphs & Algorithms II — Exercise Set 9 — HS07

URL: http://www.ti.inf.ethz.ch/ew/courses/GA07/

Exercise 26

a) Show that for any set P of n points in $\mathbb{R}^2$ there are at most $O(n^{3/2})$ pairs of points in $\binom{P}{2}$ that have Euclidean distance exactly one. (*Hint:* Show that the unit distance graph does not contain $K_{2,3}$ as a subgraph.)

b) Show that for any set P of n points in $\mathbb{R}^3$ there are at most $O(n^{5/3})$ pairs of points in $\binom{P}{2}$ that have Euclidean distance exactly one.

Homework 9

a) Let $G = (V, E)$ be a graph with average vertex degree at least $2k$. Show that there exists a subgraph H in G in which every vertex has degree at least k.

b) Let T be a tree on k vertices. Prove that $\frac{n(k-2)}{2} \leq \mathrm{ex}(n, T) < nk$, for any $n \in \mathbb{N}$ such that $n \mod (k-1) \equiv 0$.

Remark: The lower bound is conjectured to be tight.

Exercise 27

Consider the finite field $\mathcal{F}_p$ for some prime number p and define a graph $G_p = (V_p, E_p)$ on the vertex set $V_p = \mathcal{F}_p^2 \setminus \{(0,0)\}$. Two points (x, y) and (x', y') from V_p are connected by an edge in G_p if and only if they are distinct and $xx' + yy' \equiv 1 \mod p$.

a) Prove that G_p does not contain $K_{2,2}$ as a subgraph.

b) Show that $|E_p| \geq \frac{1}{2}(p-1)(p^2-1)$.

c) Conclude that $\mathrm{ex}(n, K_{2,2}) = \Omega(n^{3/2})$.

Remark: Together with Homework 10 it follows that $\mathrm{ex}(n, K_{2,2}) = \Theta(n^{3/2})$.

Homework due: 28.11.2007, 11:00AM.