

Extremal problems — Baby Examples_____

Proposition 1. If G is an n -vertex graph with **at most** $n - 2$ edges then G is disconnected.

A Question you always have to ask:

Can we improve on this proposition?

Answer. NO! The same statement is **FALSE** with $n - 1$ in the place of $n - 2$.

Proposition 1 is **best possible**, as shown by P_n .

Proposition 1. + P_n : The **minimum** value of $e(G)$ over connected graphs is $n - 1$.

Proposition 2. If G is an n -vertex graph with **at least** n edges then G contains a cycle.

Remark. Proposition 2 is also **best possible**, (e.g. P_n).

Proposition 2. + Remark: The **maximum** value of $e(G)$ over acyclic (i.e. cycle-free) graphs is $n - 1$.

Extremal problems — More Examples_____

Vague description: An **extremal problem** asks for the maximum or minimum value of a parameter over a class of objects (graphs, in most cases).

Proposition. G is an n -vertex graph with $\delta(G) \geq \lfloor n/2 \rfloor$, then G is connected.

Remark. The above proposition is **best possible**, as shown by $K_{\lfloor n/2 \rfloor} + K_{\lceil n/2 \rceil}$.

Graph $G + H$ is the **disjoint union** (or **sum**) of graphs G and H . For an integer m , mG is the graph consisting of m disjoint copies of G .

Prop. + Remark: The **maximum** value of $\delta(G)$ over disconnected graphs is $\lfloor \frac{n}{2} \rfloor - 1$.

Extremal Problems_____

graph property	graph parameter	type of extremum	value of extremum
connected	$e(G)$	minimum	$n - 1$
acyclic	$e(G)$	maximum	$n - 1$
disconnected	$\delta(G)$	maximum	$\lfloor \frac{n}{2} \rfloor - 1$
K_3 -free	$e(G)$	maximum	$\lfloor \frac{n^2}{4} \rfloor$

Triangle-free subgraphs_____

Theorem. (Mantel, 1907) The maximum number of edges in an n -vertex **triangle-free** graph is $\lfloor \frac{n^2}{4} \rfloor$.

Proof.

(i) *There is a triangle-free graph with $\lfloor \frac{n^2}{4} \rfloor$ edges.*

(ii) *If G is a triangle-free graph, then $e(G) \leq \lfloor \frac{n^2}{4} \rfloor$.*

Proof of (ii) is with extremality. (Look at the neighborhood of a vertex of maximum degree.)

Complete k -partite graphs_____

G is a **complete k -partite graph** if there is a partition $V_1 \cup \dots \cup V_k = V(G)$ of the vertex set, such that $uv \in E(G)$ iff u and v are in *different* parts of the partition. If $|V_i| = n_i$, then G is denoted by K_{n_1, \dots, n_k} .

The **Turán graph $T_{n,r}$** is the complete r -partite graph on n vertices whose parts differ in size by at most 1. (All parts have size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$.)

Lemma Among r -colorable graphs the Turán graph is the *unique* graph, which has the most number of edges.

Proof. Local change. □

Turán's Theorem

The **Turán number** $ex(n, H)$ of a graph H is the largest integer m such that there exists an H -free* graph on n vertices with m edges.

Example: Mantel's Theorem states $ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$.

Theorem. (Turán, 1941)

$$ex(n, K_r) = e(T_{n,r-1}) = \left(1 - \frac{1}{r-1}\right) \binom{n}{2} + O(n).$$

Proof. **Duplicating** a vertex v : Introduce new vertex v' , join it precisely to the neighbors of v .

Show by vertex duplication and deletion

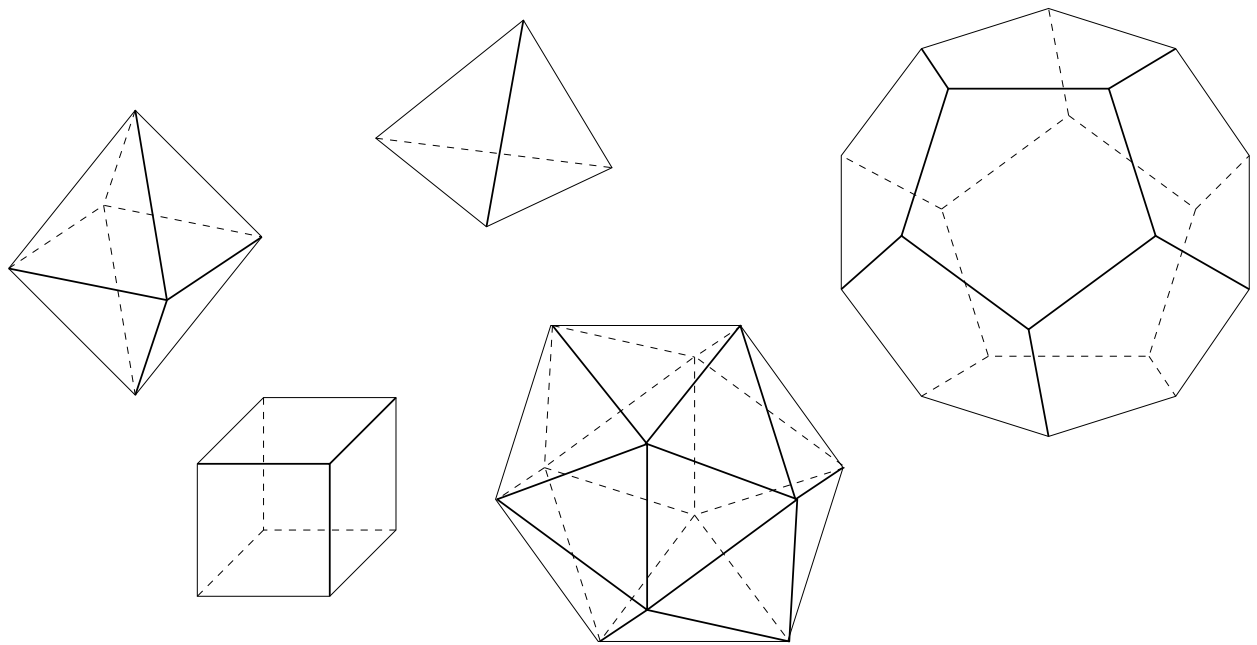
$$G \not\supseteq K_r \implies \exists \text{ complete multipartite graph } H \text{ s.t.} \\ V(H) = V(G) \text{ and } e(H) \geq e(G) \\ \text{and } K_r \not\subseteq H$$

Then apply the Lemma to finish the proof. \square

*Here H -free means that there is no subgraph isomorphic to H

Turán-type problems_____

Question. (Turán, 1941) What happens if instead of K_4 , which is the graph of the tetrahedron, we forbid the graph of some other platonic polyhedra? How many edges can a graph without an octahedron (or cube, or dodecahedron or icosahedron) have?



The platonic solids

Erdős-Simonovits-Stone Theorem_____

Theorem. (Erdős-Stone, 1946) For arbitrary fixed integers $r \geq 2$ and $t \geq 1$

$$ex(n, T_{rt,r}) = \left(1 - \frac{1}{r-1}\right) \binom{n}{2} + o(n^2).$$

Corollary. (Erdős-Simonovits, 1966) For any graph H ,

$$ex(n, H) = \left(1 - \frac{1}{\chi(H)-1}\right) \binom{n}{2} + o(n^2).$$

Corollaries of the Corollary.

$$\begin{aligned} ex(n, \text{octahedron}) &= \frac{n^2}{4} + o(n^2) \\ ex(n, \text{dodecahedron}) &= \frac{n^2}{4} + o(n^2) \\ ex(n, \text{icosahedron}) &= \frac{n^2}{3} + o(n^2) \\ ex(n, \text{cube}) &= o(n^2) \end{aligned}$$

Proof of the Erdős-Simonovits Corollary_____

Theorem. (Erdős-Stone, 1946) For arbitrary fixed integers $r \geq 2$ and $t \geq 1$

$$ex(n, T_{rt,r}) = \left(1 - \frac{1}{r-1}\right) \binom{n}{2} + o(n^2).$$

Corollary. (Erdős-Simonovits, 1966) For any graph H ,

$$ex(n, H) = \left(1 - \frac{1}{\chi(H)-1}\right) \binom{n}{2} + o(n^2).$$

Proof of the Corollary. Let $r = \chi(H)$.

- $\chi(T_{n,r-1}) < \chi(H)$, so $e(T_{n,r-1}) \leq ex(n, H)$.
- $T_{r\alpha,r} \supseteq H$, so $ex(n, T_{r\alpha,r}) \geq ex(n, H)$, where α is a constant depending on H ; say $\alpha = \alpha(H)$.

□

The number of edges in a C_4 -free graph_____

Theorem (Erdős, 1938) $ex(n, C_4) = O(n^{3/2})$

Proof. Let G be a C_4 -free graph on n vertices.

$C = C(G) :=$ number of $K_{1,2}$ (“cherries”) in G .

Doublecount C .

Counting by the midpoint: Every vertex v is the midpoint of exactly $\binom{d(v)}{2}$ cherries. Hence

$$C = \sum_{v \in V} \binom{d(v)}{2}.$$

Counting by the endpoints: Every pair $\{u, w\}$ of vertices form the endpoints of **at most one** cherry. (Otherwise there is a $C_4 \subseteq G$.) Hence

$$C \leq 1 \cdot \binom{n}{2}.$$

Proof cont'd

Combine and apply Jensen's inequality
(Note that $x \rightarrow \binom{x}{2}$ is a convex function)

$$\binom{n}{2} \geq C \geq \sum_{v \in V} \binom{d(v)}{2} \geq n \cdot \binom{\bar{d}(G)}{2}.$$

$\bar{d}(G) = \frac{1}{n} \sum_{v \in V} d(v)$ is the **average degree** of G .

$$\frac{n-1}{2} \geq \binom{\bar{d}(G)}{2} \geq \frac{(\bar{d}(G)-1)^2}{2}$$

Hence $\sqrt{n-1} + 1 \geq \bar{d}(G)$. □

Theorem (E. Klein, 1938) $ex(n, C_4) = \Theta(n^{3/2})$

Proof. Homework.

Theorem (Kővári-Sós-Turán, 1954) For $s \geq t \geq 1$

$$ex(n, K_{t,s}) \leq c_s n^{2-\frac{1}{t}}$$

Proof. Homework.

Open problems and Conjectures_____

Known results.

$$\Omega(n^{3/2}) \leq ex(n, Q_3) \leq O(n^{8/5})$$

$$\Omega(n^{9/8}) \leq ex(n, C_8) \leq O(n^{5/4})$$

$$\Omega(n^{5/3}) \leq ex(n, K_{4,4}) \leq O(n^{7/4})$$

Conjectures.

$$ex(n, Q_3) = \Theta\left(n^{8/5}\right)$$

$$ex(n, C_{2k}) = \Theta\left(n^{1+\frac{1}{k}}\right) \quad \text{true for } k = 2, 3 \text{ and } 5$$

$$ex(n, K_{t,s}) = \Theta\left(n^{2-\frac{1}{\min\{t,s\}}}\right) \quad \text{true for } t = 2, 3 \text{ and } s \geq t \\ \text{or } t \geq 4 \text{ and } s > (t-1)!$$

Alternative proof of the Erdős-Stone Thm_____

Erdős-Stone Theorem. (Reformulation) For any $\epsilon > 0$ and integers $r \geq 2, t \geq 1$ there exists an integer $M = M(r, t, \epsilon)$, such that any graph G on $n \geq M$ vertices with more than $\left(1 - \frac{1}{r-1} + \epsilon\right) \binom{n}{2}$ edges contains $T_{rt,r}$.

We derive this through the following statement.

Seemingly Weaker Theorem. For any $\epsilon > 0$ and integers $r \geq 2, t \geq 1$ there exists an integer $N = N(r, t, \epsilon)$, such that any graph G on $n \geq N$ vertices and with $\delta(G) \geq \left(1 - \frac{1}{r-1} + \epsilon\right) n$ contains $T_{rt,r}$.

Note that w.l.o.g. $\epsilon < \frac{1}{r-1}$.

Derivation of the Erdős-Stone Theorem from the Seemingly Weaker Theorem.

Let G be a graph on $n \geq M(r, t, \epsilon)^*$ vertices with more than $\left(1 - \frac{1}{r-1} + \epsilon\right) \binom{n}{2}$ edges. Recursively delete vertices which are adjacent to less than $\left(1 - \frac{1}{r-1} + \frac{\epsilon}{2}\right)$ -fraction of the remaining vertices.

What is the number n' of vertices we are left with?

We deleted at most $\sum_{j=n'+1}^n j \left(1 - \frac{1}{r-1} + \frac{\epsilon}{2}\right)$ edges. So

$$e(G) \leq \left(\binom{n+1}{2} - \binom{n'+1}{2} \right) \left(1 - \frac{1}{r-1} + \frac{\epsilon}{2}\right) + \binom{n'}{2}.$$

This implies

$$\frac{\epsilon}{2} \binom{n}{2} - n \leq \left(\frac{1}{r-1} - \frac{\epsilon}{2} \right) \binom{n'}{2} - n'.$$

We choose $M(r, t, \epsilon)$ such that $n \geq M(r, t, \epsilon)$ implies $n' \geq N(r, t, \epsilon/2)$.

*At this point we don't know $M(r, t, \epsilon)$ yet!!! We'll define it in the proof through $N(r, t, \epsilon/2)$. (which is known!)

Proof of the Seemingly Weaker Theorem.

Induction on r .

For $r = 2$ the claim is true provided $\frac{\binom{\epsilon n}{t} n}{\binom{n}{t}} > t - 1$, which is certainly true from some threshold $N(2, t, \epsilon)$.

Let $r \geq 2$ and G be a graph on $n \geq N(r + 1, t, \epsilon)^*$ vertices with $\delta(G) \geq \left(1 - \frac{1}{r} + \epsilon\right) n$.

We would like to find a $T_{(r+1)t, r+1}$ in G .

Let $s = \left\lceil \frac{t}{\epsilon} \right\rceil$. By the induction hypothesis[†] there is a $T_{rs, r}$ in G with vertex-set $A_1 \cup \dots \cup A_r$, where $|A_1| = \dots = |A_r| = s$.

$$U = V(G) \setminus (A_1 \cup \dots \cup A_r).$$

$W = \{w \in U : |N(w) \cap A_i| \geq t, i = 1, \dots, r\}$ is the set of vertices eligible to extend some part of A_1, \dots, A_r into a $T_{(r+1)t, r+1}$.

*Again, we don't know $N(r + 1, t, \epsilon)$ yet.

†Here we assume $N(r + 1, t, \epsilon) \geq N(r, s, \epsilon)$.

Double-count the number of edges missing between U and $A_1 \cup \dots \cup A_r$. They are

- at least $(|U| - |W|)(s - t)$ and
- at most $rs \left(\frac{1}{r} - \epsilon\right) n$.

From this we have

$$|W| \geq \frac{(r - 1)\epsilon}{1 - \epsilon} n - rs$$

Thus if n is large enough* then

$$|W| > \binom{s}{t}^r (t - 1).$$

So we can select t vertices from W , which are adjacent to the same t vertices in each A_i .

*If $N(r + 1, t, \epsilon) > \left(\binom{s}{t}^r (t - 1) + rs\right) \frac{1 - \epsilon}{(r - 1)\epsilon}$