RECALL: Vertex coloring, chromatic number

A k-coloring of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$. The labels are called colors; all vertices that receive the same color form a color class.

A k-coloring is proper if adjacent vertices have different labels. A graph is k-colorable if it has a proper k-coloring.

The chromatic number is

$$\chi(G) := \min\{k : \ G \text{ is } k\text{-colorable}\}.$$

A graph G is k-chromatic if $\chi(G) = k$. A proper k-coloring of a k-chromatic graph is an optimal coloring.

Examples. K_n, $K_{n,m}$, C_5, Petersen

RECALL: Lower bounds

Simple lower bounds

$$\begin{aligned} \chi(G) &\geq \omega(G) \\ \chi(G) &\geq \frac{v(G)}{\alpha(G)} \end{aligned}$$

Examples for $\chi(G) \neq \omega(G)$:

- odd cycles of length at least 5,

$$\chi(C_{2k+1}) = 3 > 2 = \omega(C_{2k+1})$$

- complements of odd cycles of order at least 5,

$$\chi(\overline{C}_{2k+1}) = k + 1 > k = \omega(\overline{C}_{2k+1})$$

- random graph $G = G(n, \frac{1}{2})$, almost surely

$$\chi(G) \approx \frac{n}{2 \log n} > 2 \log n \approx \omega(G)$$

RECALL: Upper bounds

Proposition $\chi(G) \leq \Delta(G) + 1$.

Proof. Algorithmic; Greedy coloring.

A graph G is d-degenerate if every subgraph of G has minimum degree at most d.

> **Claim.** G is d-degenerate iff there is an ordering of the vertices $v_1, \dots, v_n$, such that $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| \leq d$

Proposition. For a d-degenerate G, $\chi(G) \leq d + 1$. In particular, for every G, $\chi(G) \leq \max_{H \subseteq G} \delta(H) + 1$.

Proof. Greedy coloring.

Brooks' Theorem. (1941) Let G be a connected graph. Then $\chi(G) = \Delta(G) + 1$ iff G is a complete graph or an odd cycle.

Proof. Trickier, but still greedy coloring...

Equitable colorings

Definition A coloring of G is equitable if it is proper and the size of any two color classes differ by at most one.

Applications Many...

Conjecture (Erdős, 1964) For each $r \geq \Delta(G)$, G has an equitable $(r+1)$-coloring.

Remark Strengthening of the greedy coloring upper bound $\chi(G) \leq \Delta(G) + 1$.

Theorem (Hajnal-Szemerédi, 1970) For each $r \geq \Delta(G)$, G has an equitable $(r+1)$-coloring.

Proof. Complicated, long (22 pages).

Question Is there a polynomial time algorithm which, given a graph G, finds an equitable $(\Delta(G)+1)$-coloring of G?

New (2006) proof by Kierstead and Kostochka comes with a polynomial time algorithm.

Constructions, remarks, special cases

Let n be even.
What does the HSz-Theorem say if $\Delta(G) \leq \frac{n}{2} - 1$?

There is an equitable $\frac{n}{2}$-coloring of G.

Or equivalently: In $\bar{G}$ there is a perfect matching.

Special case 1: of HSzT:
$\delta(H) \geq \frac{n}{2} \ \Rightarrow\ H$ has a perfect matching.

In fact, even more is true: **Dirac's Theorem**:
$\delta(H) \geq \frac{n}{2} \ \Rightarrow\ H$ has a Hamilton cycle!

Special case 2: (Corrádi-Hajnal Theorem)
If $3|n$ and $\delta(H) \geq \frac{2n}{3}$, then H has a K_3-factor (a family of triangles partitioning the vertex set).

Nearly equitable colorings

$|V(G)| = s(r+1)$, where $r \geq \Delta(G)$.

An $(r+1)$-coloring f of G is nearly equitable if it is proper and all classes have the same size s except for one small class $V^- = V^-(f)$ with size $s-1$ and one large class $V^+ = V^+(f)$ with size $s+1$.

Let U and W be two distinct color classes of f. Vertex $y \in U$ is movable to W if y has no neighbors in W.

Auxiliary digraph $H = H(G, f)$.
$V(H) = \{U : U \text{ is a color class of } f\}$.
$UW \in E(H)$ if some vertex of U is movable to W.

W is accessible if there is a WV^--path in H.
Remark: V^- is accessible.

$\mathcal{A} = \mathcal{A}(f)$ the family of accessible classes.

Accessible classes — how they can help

$A := \cup\mathcal{A}$, $B := V(G) \setminus A$

Lemma 1: If G has a nearly equitable $(r+1)$-coloring f whose large class $V^+ \in \mathcal{A}$, then G has an equitable $(r+1)$-coloring.

Hence assume $V^+ \notin \mathcal{A}$

$m := |\mathcal{A}| - 1$

$q := r - m$ the number of non-accessible classes

Facts: $|A| = (m+1)s - 1$

$|B| = qs + 1$

$y \in B$ cannot be moved to A, so
$d_A(y) \geq m + 1$, which implies $d_B(y) \leq q - 1$.

Consequence: Kicking out any vertex y from B leaves us with a subgraph $H = G[B \setminus \{y\}]$ having qs vertices and $\Delta(H) \leq q - 1$. An invitation for induction!

Terminal classes

$m \geq 1$, since otherwise $A = V^-$, so

$$
\begin{aligned}
rs + 1 = |B| \leq& \\
\sum_{y \in B} d_A(y) = \ & |E(A,B)| = \sum_{x \in A} d_B(x) \\
& \leq r|V^-| = r(s-1)
\end{aligned}
$$

For $W \in \mathcal{A}$ let $\mathcal{A} \setminus \{W\} = \mathcal{S}_W \cup \mathcal{T}_W$, where

$\mathcal{T}_W = \{Z \in \mathcal{A}$: every ZV^--path goes through $W\}$.

$U \in \mathcal{A}$ is terminal if $\mathcal{T}_U = \emptyset$, that is,
there is a ZV^--path avoiding U for each $Z \in \mathcal{A} \setminus \{U\}$.
U is non-terminal if $\mathcal{T}_U \neq \emptyset$.

Remark: V^- is non-terminal

$$\mathcal{S}_{V^-} = \emptyset, \quad \mathcal{T}_{V^-} = \mathcal{A} \setminus \{V^-\}.$$

Fix a non-terminal class $U \in \mathcal{A}$ and let $\mathcal{A}' := \mathcal{T}_U$.

Solo vertices

$t := |\mathcal{A}'|$, $A' := \cup\mathcal{A}'$

$x \in A' \Rightarrow x$ is not movable to any class in $\mathcal{A} \setminus \mathcal{A}' \setminus \{U\}$

$$\Rightarrow d_A(x) \geq (m+1) - (t+1) = m - t$$

zy is a solo edge and z and y are solo vertices if

- $y \in B$
- $z \in W \in \mathcal{A}'$
- $N_W(y) = \{z\}$

Remark: y is movable to $W \setminus \{z\}$.

$S_z := \{y \in B : zy$ is a solo edge$\}$

$S^y := \{z \in A' : zy$ is a solo edge$\}$

Claim: $y \in B \Rightarrow |S^y| \geq t - q + 1 + d_B(y)$.

Proof: $t - |S^y| \leq |\{Z \in \mathcal{A} : |Z \cap N(y)| \geq 2\}|$

$$\leq r - d_B(y) - (m+1)$$

Lemmas

Lemma 2: If there exists $W \in \mathcal{A}'$ such that no solo vertex in W is movable to a class in $\mathcal{A} \setminus \{W\}$ then $q + 1 \leq t$. Furthermore, every vertex of B is solo.

Remark: For any solo vertex $z \in W$ there is a $y \in B$ which we could move to W — should we be able to get rid of z by moving it further. Lemma 2 discusses the “bad case”, when this is not possible for any solo vertex z.

Lemma 3: Let $W \in \mathcal{A}'$. Then $\exists$ a solo vertex $z \in W$ such that either z is movable to a class in $\mathcal{A} \setminus \{W\}$
or S_z is *not* a clique.

Proof of Lemma 2

Lemma 2: If there exists $W \in \mathcal{A}'$ such that no solo vertex in W is movable to a class in $\mathcal{A} \setminus \{W\}$ then $q + 1 \leq t$. Furthermore, every vertex of B is solo.

Proof: Doublecount $|E(W, B)|$.

$S \subseteq W$ set of solo vertices in W, $D = W \setminus S$.

No $z \in S$ is movable to $\mathcal{A} \setminus \{W\} \Rightarrow d_A(z) \geq m$

$$\Rightarrow d_B(z) \leq q$$

No $z \in D$ is movable to $\mathcal{A} \setminus \mathcal{A}' \setminus \{U\} \Rightarrow d_B(z) \leq q + t$

$$|N_B(S)| + 2(|B| - |N_B(S)|) \leq |E(W, B)|$$
$$\leq q|S| + (t + q)|D|$$

$$\begin{aligned} 2(qs + 1) - q|S| &\leq qs + t|D| \\ q + \frac{2}{|D|} &\leq t \end{aligned}$$

By Claim $|S^y| \geq t - q + 1 + d_B(y) \geq 2$. $\square$

Proof of Lemma 3

Lemma 3: Let $W \in \mathcal{A}'$. Then $\exists$ a solo vertex $z \in W$ such that either z is movable to a class in $\mathcal{A} \setminus \{W\}$
or S_z is *not* a clique.

Proof: Suppose the statement is false.
Lemma 2 $\Rightarrow \forall y \in B$ is solo and $t - q \geq 1$.
S_z is a clique $\Rightarrow \forall y \in S_z$, $d_B(y) + 1 \geq |S_z|$.*

$$\mu(xy) = \begin{cases} \frac{q}{|S_x|} & \text{if } xy \text{ is solo} \\ 0 & \text{otherwise} \end{cases}$$

Doublecount $\mu(A', B) = \sum_{x \in A'} \sum_{y \in B} \mu(xy)$.

$$\sum_{x \in A'} \sum_{y \in B} \mu(xy) = \sum_{\text{solo } z \in A'} |S_z| \cdot \frac{q}{|S_z|} \leq qst.$$

$$\begin{aligned} \sum_{y \in B} \sum_{x \in A'} \mu(xy) &= \sum_{y \in B} \sum_{z \in S^y} \frac{q}{|S_z|} \geq \sum_{y \in B} |S^y| \frac{q}{c_y} \\ &\geq \sum_{y \in B} (t - q + c_y) \frac{q}{c_y} \\ &\geq \sum_{y \in B} t = t|B| = t(qs + 1) \end{aligned}$$

$^*c_y = \max\{|S_z| : z \in S^y\} \leq d_B(y) + 1 \leq q$

Proof of the Hajnal-Szemerédi Theorem

Theorem (Hajnal-Szemerédi, 1970) For each $r \geq \Delta(G)$, G has an equitable $(r+1)$-coloring.

Proof:

WLOG $n = s(r+1)$. Let G be a counterexample on n vertices with the smallest number of edges.

Consequence: For arbitrary edge $e = xy \in E(G)$,

- there is an equitable $(r+1)$-coloring f_0 of $G - e$
- x and y must be in the same color class V of f_0.

$d(x) \leq r \quad \Rightarrow \quad \exists$ class $W \neq V$, x is movable to W.
$\Rightarrow \quad \exists$ a nearly equitable $(r+1)$-coloring of G.

Let f be a nearly equitable $(r+1)$-coloring of G such that the number $q = q(f)$ of nonaccessible classes is minimal.

We can assume by Lemma 1 that $V^+ \notin \mathcal{A} = \mathcal{A}(f)$.

Proof of Hajnal-Szemerédi Theorem – cont'd

Fix non-terminal class $U \in \mathcal{A}$ with $\mathcal{T}_U =: \mathcal{A}'$ minimal.
Recall: $\mathcal{A}' \neq \emptyset$
Minimality $\Rightarrow$ every class in $\mathcal{A}'$ is terminal

Lemma 3 $\Rightarrow$ $\exists$ class $W \in \mathcal{A}'$, a solo vertex $z \in W$ and a vertex $y_1 \in S_z$ such that

- either z is movable to a class $X \in \mathcal{A} \setminus \{W\}$
- or z is not movable in $\mathcal{A}$ and there exists another vertex $y_2 \in S_z$ which is not incident to y_1.

Recall: $A = \cup\mathcal{A}$, $B = V(G) \setminus A$

Define: $A^+ := A \cup \{y_1\}$, $B^- := B \setminus \{y_1\}$

Recall: there exists an equitable q-coloring g of $G[B^-]$

($\forall y \in B$, $d_B(y) \leq q - 1 \Rightarrow$ induction applies)

Proof of Hajnal-Szemerédi Theorem – Cases

Case 1. z is movable to $X \in \mathcal{A}$.

W is terminal $\Rightarrow \exists\, XV^{-}$-path in H avoiding W.
Move vertices along this path, z to X, y_1 to $W \setminus \{z\}$
This creates an equitable $(m+1)$-coloring φ' of $G[A^{+}]$.
Then $\varphi' \cup g$ is an equitable $(r+1)$-coloring of G,
a **contradiction**.

Case 2. z is not movable to any class in $\mathcal{A}$.

Then $d_{B^-}(z) \leq q-1$ and z can be moved into a color class $Y \subseteq B^-$ of g. This defines a new equitable q-coloring g' of G on $B^* = B^- \cup \{z\}$.
Move y_1 to $W \setminus \{z\}$ to obtain an equitable $(m+1)$-coloring ψ of G on $A^* = V(G) \setminus B^*$.
$\psi' := \psi \cup g'$ is a nearly equitable $(r+1)$-coloring of G.

$A^* \subseteq A(\psi')$ and the class of y_2 is now accessible!
(since y_2 is movable to $W^* = W \cup \{y_1\} \setminus \{z\}$)

Thus $q(\psi') < q(f)$, a **contradiction**. $\square$

Fast equitable coloring

Algorithm `EquiColor`**(G, r)** (Kostochka-Kierstead, 2006)

Input: graph G, integer $r \geq \Delta(G)$

Output: equitable $(r+1)$-coloring of G

$V(G) = \{v_1, \ldots, v_n\}$, $n = s(r+1) - p$
For $i, 0 \leq i < n$, let $G_i \subseteq G$, $V(G_i) = V(G)$,
$$E(G_i) = \{xy : x \text{ or } y = v_j, j \leq i\}$$

```
IF $p \neq 0$ THEN
    output EquiColor($G + K_p, r$)$|_{V(G)}$
ELSE
$i := 0$, $f_0 :=$ arbitrary equitable $(r+1)$-coloring of $G_0$
    WHILE $i < n - 1$ DO $i := i + 1$
    IF $v_i$ has no $G$-neighbors in its $f_{i-1}$-color class THEN
    $f_i := f_{i-1}$
    ELSE Define $f'_{i-1}$ from $f_{i-1}$ by moving $v_i$ to
        an $f_{i-1}$-color class that has no $G$-neighbors of $v_i$
        $f_i :=$ Equitizer($G_i, f'_{i-1}$)
output $f_{n-1}$.
```

Algorithm `Equitizer`(G, f)

Input: graph G on $n := s(r+1)$ vertices, $\Delta(G) \leq r$;
nearly equitable $(r+1)$-coloring f of G

Output: equitable $(r+1)$-coloring f' of G

```
Construct auxiliary digraph $H = H(f)$
IF $V^+ \in \mathcal{A}$ THEN
    $f'$ := recoloring of $f$ by moving vertices
            along a $V^+V^-$-path in $H$
ELSE
    Construct $\mathcal{A}, \mathcal{A}', B, W, z, y_1$ as in the proof
    $g$ := Equitizer($G[B^-], f|_{B^-}$)
    IF $z$ is movable to an $X \in \mathcal{A}$ (i.e., Case 1) THEN
        construct $\varphi'$;   $f'$ := $\varphi' \cup g$
    ELSE (i.e., Case 2)
        find $y_2 \in S_z$ such that $y_1y_2 \notin E$
        construct $g'$ on $B^*$ and $\phi$ on $A^*$
        construct nearly equitable $\psi'$ := $\psi \cup g'$
        $f'$ := Equitizer($G, \psi'$)   [choose $\mathcal{A}'(\psi') \subseteq B(f)$]
output $f'$
```

Running time

Theorem: `EquiColor`(G, r) outputs an equitable $(r+1)$-coloring of G in time $O(n^5)$.

Theorem: There exists a constant c such that algorithm `Equitizer`(G, f) outputs an equitable $(r+1)$-coloring of G in time $c(q+1)n^3$, where $q = q(f)$ is the number of non-accessible classes of f.

Corollary: Algorithm `EquiColor`(G, r) constructs an equitable $(r+1)$-coloring of graph G in time $O(n^5)$.

Running time analysis

Proof of Theorem. Termination of `Equitizer` is clear.

$R(G, f)$:= runtime of `Equitizer`(G, f)
$R(n, q)$:= max$\{R(G, f) : |V(G)| = n,\ q(f) \le q\}$

Let c be a constant such that all of the lines of `Equitizer` not calling itself recursively (i.e. recolorings, searches, constructions, case-determinations) can be performed in time $\frac{c}{2}n^3$.

Induction on $q = q(f)$:

$q = 0 \;\; \Rightarrow \;\; V^+ \in \mathcal{A}$ and $R(n, 0) \le cn^3$.

Assume $q > 0$

If Case 1 happens:

$$\begin{aligned} R(n, q) &\le \tfrac{c}{2}n^3 + R(|B^-|, q-1) \le cn^3 + cqn^3 \\ &\le c(q+1)n^3 \end{aligned}$$

Running time analysis — cont'd

If Case 2 happens: $q(\psi') < q$ and

$$R(n,q) \le \frac{c}{2}n^3 + R(|B^-|, q-1) + R(G,\psi')$$

Case 2 $\Rightarrow$ Lemma 2 $\Rightarrow$ $q+1 \le t$ $\Rightarrow$ $|B^-| \le \frac{n}{2}$

$$\Rightarrow R(|B^-|, q-1) \le cq\left(\frac{n}{2}\right)^3$$

Using $R(G,\psi') \le cqn^3$ would not suffice.
We go one deeper into the algorithm.

If, after Case 2, Case 1 happens:

Because $|B^-(\psi')| \le |B^-(f)|$, we have

$$\begin{aligned} R(G,\psi') &\le \frac{c}{2}n^3 + R(|B^-(\psi')|, q(\psi')) \\ &\le \frac{c}{2}n^3 + cq\left(\frac{n}{2}\right)^3 \end{aligned}$$

If, after Case 2, Case 2 happens again:

$\mathcal{A}'(\psi') \subseteq B(f)$ $\Rightarrow$ $q(\psi') + t(\psi') \le q(f)$
Lemma 2 $\Rightarrow$ $q(\psi') + 1 \le t(\psi')$
$\Rightarrow$ $q(\psi') \le \frac{q(f)-1}{2}$.

$$R(G,\psi') \le R(n,(q-1)/2) \le c\frac{q+1}{2}n^3 \qquad \square$$