When is a graph planar?

Theorem(Euler, 1758) If a plane multigraph G with k components has n vertices, e edges, and f faces, then

$$n - e + f = 1 + k.$$

Corollary If G is a simple, planar graph with $n(G) \geq 3$, then $e(G) \leq 3n(G) - 6$.
If G is also triangle-free, then $e(G) \leq 2n(G) - 4$.

Corollary K_5 and $K_{3,3}$ are non-planar.

The subdivision of edge $e = xy$ is the replacement of e with a new vertex z and two new edges xz and zy. The graph H' is a subdivision of H, if one can obtain H' from H by a series of edge subdivisions. Vertices of H' with degree at least three are called branch vertices.

Theorem(Kuratowski, 1930) A graph G is planar iff G does not contain a subdivision of K_5 or $K_{3,3}$.

Kuratowski's Theorem

Theorem(Kuratowski, 1930) A graph G is planar iff G does not contain a subdivision of K_5 or $K_{3,3}$.

Proof.

A Kuratowski subgraph of G is a subgraph of G that is a subdivision of K_5 or $K_{3,3}$. A minimal nonplanar graph is a nonplanar graph such that every proper subgraph is planar.

A counterexample to Kuratowski's Theorem constitutes a nonplanar graph that does not contain any Kuratowski subgraph.

Kuratowski's Theorem follows from the following Main Lemma and Theorem.

The spine of the proof

Main Lemma. If G is a graph with fewest edges among counterexamples, then G is 3-connected.

> **Lemma 1.** Every minimal nonplanar graph is 2-connected.
>
> **Lemma 2.** Let $S = \{x, y\}$ be a separating set of G. If G is a nonplanar graph, then adding the edge xy to some S-lobe of G yields a nonplanar graph.

Main Theorem.(Tutte, 1960) If G is a 3-connected graph with no Kuratowski subgraph, then G has a convex embedding in the plane with no three vertices on a line.

> A convex embedding of a graph is a planar embedding in which each face boundary is a convex polygon.
>
> **Lemma 3.** If G is a 3-connected graph with $n(G) \geq 5$, then there is an edge $e \in E(G)$ such that $G \cdot e$ is 3-connected.
>
> > *Notation:* $G \cdot e$ denotes the graph obtained from G after the contraction of edge e.
>
> **Lemma 4.** G has no Kuratowski subgraph $\Rightarrow$ $G \cdot e$ has no Kuratowski subgraph.

Proof of Tutte's Theorem

Main Theorem. (Tutte, 1960) If G is a 3-connected graph with no Kuratowski subgraph, then G has a convex embedding in the plane with no three vertices on a line.

Proof. Induction on $n(G)$.
Base case: G is 3-connected, $n(G) = 4 \Rightarrow K_4$.

Let $e \in E$ be s.t. $H = G \cdot e$ is 3-connected. (Lemma 3)
Then H has no Kuratowski subgraph. (Lemma 4)
Induction $\Rightarrow$ H has a convex embedding in the plane with no three vertices on a line.

Let $z \in V(H)$ be the contracted e.
$H - z$ is 2-connected $\Rightarrow$ boundary of the face containing z after the deletion of the edges incident to z is a cycle C.

Let $x_0, \ldots, x_{k-1}$ be the neighbors of x on C in cyclic order. Note that $|N(x)| \geq 3$ and hence $k \geq 1$.

Denote by $\langle x_i, x_{i+1} \rangle$ the portion of C from x_i to x_{i+1} (including endpoints; indices taken modulo k.)

Let $N_x = N(x) \setminus \{y\}$ and $N_y = N(y) \setminus \{x\}$.

Case 1. $|N_x \cap N_y| \geq 3$.

Let $u, v, w \in N_x \cap N_y$. Then x, y, u, v, w are the branch vertices of a K_5-subdivision in G.

Case 2. $|N_x \cap N_y| \leq 2$.

Since $|N_x \cup N_y| \geq 3$, there is w.l.o.g. a vertex $u \in N_y \setminus N_x$. Let i be such that u is on $\langle x_i, x_{i+1} \rangle$.

Case 2a. N_y is contained in $\langle x_i, x_{i+1} \rangle$.

Then there is an appropriate embedding of G: Placing x in place of z and y sufficiently close to x maintains convexity. (No three vertices are collinear).

Case 2b. For every i there is a vertex in N_y that is not contained in $\langle x_i, x_{i+1} \rangle$.

Then there must be a $v \in N_y$ that is not on $\langle x_i, x_{i+1} \rangle$ and x, y, x_i, x_{i+1}, u, v are the branch vertices of a $K_{3,3}$-subdivision in G.

Proof of the Lemmas

Lemma 3. G is 3-connected, $n(G) \geq 5 \Rightarrow$ there is an edge $e \in E(G)$ such that $G \cdot e$ is 3-connected.

Proof. Suppose G is 3-connected and for every $e \in E$, $G \cdot e$ is NOT 3-connected.

For edge $e = xy$, the vertex z is a mate of xy if $\{x, y, z\}$ is a cut in G.

Choose $e = xy$ and their mate z such that $G - \{x, y, z\}$ has a component H, whose order is as large as possible.

Let H' be another component of $G - \{x, y, z\}$ and let $u \in V(H')$ be a neighbor of z (there is one). Let v be a mate of uz.

$V(H) \cup \{x, y\} \setminus \{v\}$ is connected in $G - \{z, u, v\}$ contradicting the maximality of H. □

Lemma 4. G has no Kuratowski subgraph $\Rightarrow$ $G \cdot e$ has no Kuratowski subgraph.

Proof. Suppose $G \cdot e$ contains a Kuratowski subgraph H. Then

- $z \in V(H)$
- z is a branchvertex of H
- $|N_H(z)| = 4$ and $|N_H(z) \cap N_G(x)|, |N_H(z) \cap N_G(y)| \geq 2$

Then H is the subdivision of K_5 $\Rightarrow$ G contains a subdivision of $K_{3,3}$, a contradiction. □

Minors

K_7 is a toroidal graph (it is embeddable on the torus), K_8 is not. What else is not? For the torus there is NO equivalent version of Kuratowski's characterization with a finite number of forbidden subdivisions. Any such characterization must lead to an infinite list.

A weaker concept: Minors.

Graph G is called a minor of graph H if G can be obtained from H by a series of edge deletions and edge contractions. Graph H is also called a G-minor

Example: K_5 is a minor of the Petersen graph P, but P does not contain a K_5-subdivision.

The Graph Minor Theorem

Theorem. (Robertson and Seymour, 1985-2005) In any infinite list of graphs, some graph is a minor of another.

Proof: more than 500 pages in 20 papers.

Corollary For any graph property that is closed under taking minors, there exists finitely many minimal forbidden minors.

Remark: Wagner's Theorem, stating that every non-planar graph contains either a K_5 or $K_{3,3}$-minor, can be (quite straightforwardly) deduced from Kuratowski's Theorem.

For embeddability on the projective plane, it is known that there are 35 minimal forbidden minors. For embeddability on the torus, we don't know the exact number of minimal forbidden minors; there are more than 800 known.