Nowhere zero flow

Definition: A flow on a graph $G = (V, E)$ is a pair (D, f) such that

1. D is an orientation of G.
2. f is a function on E.
3. $\sum_{u \in N_D^+(v)} f(uv) = \sum_{w \in N_D^-(v)} f(vw)$ for every $v \in V$.

Example: $f \equiv 0$.

Definition: A nowhere zero flow on a graph $G = (V, E)$ is a flow (D, f) such that $f(e) \neq 0$ for every $e \in E$.

Definition: For a positive integer k, a k-flow on a graph $G = (V, E)$ is a flow (D, f) such that $f : E \to \mathbb{Z}$ and $-(k-1) \leq f(e) \leq k-1$ for every $e \in E$.

Definition: A flow (D, f) on $G = (V, E)$ is called positive if $f(e) > 0$ for every $e \in E$.

Definition: A k-flow which is nowhere zero is called a k-nowhere zero flow or k-NZF for brevity.

Proposition: The following conditions are equivalent:

1. G admits a positive k-flow.

2. G admits a k-NZF.

3. Every orientation of G admits a k-NZF.

Corollary: Admiting a k-NZF is a property of the underlying undirected graph.

Which graphs admit a k-NZF?

Theorem: A graph G admits a k-NZF (for some k that might depend on G) iff it is bridgeless.

Lemma: The flow along any (directed) cut is zero.

Proof (of the Lemma and the Theorem): Exercise.

Question: Is there a fixed k such that every bridgeless graph admits a k-NZF? We will answer this question later.

Small values of k:

Observation: No graph admits a 1-NZF. A graph admits a 2-NZF iff it is even.

Proposition: (Tutte 1949) A cubic graph admits a 3-NZF iff it is bipartite.

Flow coloring duality

Definition: A k-coloring of a graph $G = (V, E)$ is a mapping $f : V \to \{0, 1, \ldots, k-1\}$. A coloring f is proper if $f(u) \neq f(v)$ for every edge $(u, v) \in E$.

Remark: A graph $G = (V, E)$ admits a k-coloring (for some k that might depend on G) iff it is loopless. Moreover, a coloring $f : V \to \{0, \ldots, k-1\}$ induces a function $g : E \to \{0, \ldots, k-1\}$ by setting $g(u, v) = |f(u) - f(v)|$. Note that g is nowhere zero iff f is proper. Moreover, g vanishes along (directed) cycles.

Definition: Let $G = (V, E)$ be a plane graph. The dual of G is a graph $G^* = (V^*, E^*)$, where V^* is the set of faces of G, and $(f_1, f_2) \in E^*$ iff they share an edge in G. Note that, in some sense, $E^* = E$.

Observation: If $e \in E$ is a bridge, then $e^* \in E^*$ is a loop. If $e_1, \ldots, e_k \in E$ are the edges of a circuit of G, then $e_1^*, \ldots, e_k^* \in E$ are the edges of a cut of G^*. This can be (somehow) generalized to non-planar graphs.

Flows and colorings of plane graphs

Theorem: (Tutte 1954) A plane bridgeless graph $G = (V, E)$ is k-face-colorable (its dual is k-colorable) iff it admits a k-NZF.

proof: Let c be a proper k-coloring of the faces of G. Let D be an arbitrary orientation of G. Define a function $f : E \to \mathbb{Z}$ by setting $f(e) = c(f_L^e) - c(f_R^e)$, where f_L^e is the face to the left of e (w.r.t. to its direction under D) and f_R^e is the face to the right of e.

Conversely, let (D, f) be a k-NZF on G. Define a k-coloring c of the faces of G as follows. Color the unbounded face with color 0. For any other face F, traverse faces (in an arbitrary route) to the outer face. Define $c(F)$ to be the (directed) sum of flows through edges you cross. Add $f(e)$ if e points to your right and subtract otherwise. All calculations are modulo k. Note that c is a well-defined proper k-coloring.

Group flow

Definition: Let Γ be an abelian group (say, w.r.t. addition). A Γ-flow on a graph $G = (V, E)$ is a flow $f : E \to \Gamma$. A Γ-flow f, such that $f(e) \neq 0$ (that is, $f(e)$ is not the identity element of Γ) for every $e \in E$, is called a Γ nowhere zero flow or Γ-NZF for brevity.

Theorem: (Tutte 1954) Let Γ be a group of order k and let G be a graph. Then, G admits a k-NZF iff G admits a Γ-NZF.

Theorem: (Tait 1878) A simple bridgeless cubic plane graph is 3-edge colorable iff it is 4-face colorable.

Theorem: A simple bridgeless cubic graph is 3-edge colorable iff it admits a 4-NZF.

proof: Use the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Tutte's flow conjectures

Conjecture: (3-NZF) Every 4-edge-connected graph admits a 3-NZF (dual to Grötzsch's Theorem).

Conjecture: (4-NZF) Every bridgeless graph containing no subdivision of the Petersen graph admits a 4-NZF (dual to the 4-color Theorem).

Conjecture: (5-NZF) Every bridgeless graph admits a 5-NZF.

> weaker version: There exists a positive integer k such that every bridgeless graph admits a k-NZF.

4-NZF in 4-edge-connected graphs

Theorem: Every 4-edge-connected graph admits a 4-NZF.

Proof: Let $G = (V, E)$ be a 4-edge-connected graph. Let T_1, T_2 be two edge disjoint spanning trees of G (Tutte 1961, Nash-Williams 1961).

For $i = 1, 2$ and for every $e \in E \backslash E(T_i)$, let C_e^i denote the unique circuit in $T_i \cup \{e\}$, and let f_e^i be a flow with values $\pm i$ on the edges of C_e^i and 0 elsewhere.

Let $f_1 = \sum_{e \in E \backslash E(T_1)} f_e^1 \mod 4$. Note that $f_1(e) = \pm 1$ for every $e \in E \setminus E(T_1)$.

Let $F = \{e \in E(T_1) : f_1(e) = 0 \mod 4\}$, let $f_2 = \sum_{e \in F} f_e^2 \mod 4$, and let $f = f_1 + f_2 \mod 4$.

Note that f is a $\mathbb{Z}_4$-NZF.

Solving the weak 5-NZF Conjecture

Theorem: (Kilpatrick 1975, Jaeger 1979) Every bridgeless graph admits an 8-NZF.

Theorem: (Seymour 1981) Every bridgeless graph $G = (V, E)$ admits a 6-NZF.

Proof: Main idea - find a "good" even subgraph H of G. Prove that $G \setminus H$ admits a $\mathbb{Z}_3$-NZF f. Let g be a $\mathbb{Z}_2$-NZF on H; then (f, g) is a $(\mathbb{Z}_3 \times \mathbb{Z}_2)$-NZF on G.

Construction of a good H: We construct a series of graphs $H^0 \subseteq \ldots \subseteq H^r$, where r is the smallest index i such that H^i spans G, recursively - H^0 is an arbitrary vertex of V, and $H^i = (H^{i-1} \cup H_i) + F_i$ for every $1 \leq i \leq r$.

Definition of H_i and F_i: Let $X_i \subseteq V \setminus V(H^{i-1})$ be minimal such that $X_i \neq \emptyset$ and $e(X_i, V \setminus (V(H^{i-1}) \cup X_i)) \leq 1$. Let F_i be an arbitrary pair of edges connecting X_i and $V(H^{i-1})$ (if there is only one such edge, then take it). Let H_i be any even connected subgraph of $G[X_i]$ that contains the "X_i-endpoints" of the edges in F_i.

Observations:

1. X_i is well defined (as $X_i = V \setminus V(H^{i-1})$ is valid).
2. $e(X_i, V(H^{i-1})) \geq 1$ (as $e(X_i, V \setminus (V(H^{i-1}) \cup X_i)) \leq 1$ and G is 2-edge-connected).
3. $G[X_i]$ is 2-edge-connected or a single vertex (as X_i is minimal).
4. H_i is well defined ($G[X_i]$ is 2-edge-connected, Menger).

Let $G' = G \setminus H^r = (V, E')$ and let $H = H^0 \cup \bigcup_{i=1}^r H_i$. Note that H is even and that H^r is spanning and connected.

Definition of a $\mathbb{Z}_3$-flow:

We define $\mathbb{Z}_3$-flows $f_r, f_{r-1}, \ldots, f_0$ in "reverse" induction, such that f_i is nowhere zero on $E' \cup \bigcup_{j=i+1}^{r} F_j$, for every $0 \le i \le r$.

Induction base: For every $e \in E'$, let C_e denote an arbitrary circuit in $H^r \cup \{e\}$ that contains e, and let f_e be a flow with values ± 1 on the edges of C_e and 0 elsewhere. Let $f_r = \sum_{e \in E'} f_e$.

Induction step: $f_r, \ldots, f_i \Rightarrow f_{i-1}$

Case 1: $|F_i| = 1$, say $F_i = \{e\}$. Let $f_{i-1} = f_i$. We claim that $f_i(e) \neq 0$. It suffices to prove that $f_i(e') \neq 0$, where e' is the unique edge in $E(X_i, V \setminus (V(H^{i-1}) \cup X_i))$. By the induction hypothesis, it suffices to show that $e' \in E' \cup \bigcup_{j=i+1}^{r} F_j$. Clearly, $e' \notin \bigcup_{j=1}^{i} F_j$ and $e' \notin \bigcup_{k=1}^{i-1} H_k$. Moreover, e' is a bridge of $G[V \setminus V(H^{i-1})]$ and for every $i \le k \le r$, H_k is a bridgeless subgraph of $G[V \setminus V(H^{i-1})]$.

Case 2: $|F_i| = 2$, say $F_i = \{e_1, e_2\}$. Since H^{i-1} and H_i are both connected, there exists a cycle C in $(H^{i-1} \cup H_i) + F_i$ that contains e_1 and e_2. If $f_i(e_1) \neq 0$ and $f_i(e_2) \neq 0$, then let $f_{i-1} = f_i$. Otherwise Assume wlog that $f_i(e_1) = 0$. Circulate a flow of 1 on C in the direction agreeing with e_2; denote this flow by g. Let $f_{i-1} = f_i + g$. Note that $f_{i-1}(e_1) \neq 0$, $f_{i-1}(e_2) \neq 0$ and $f_{i-1}(e) = f_i(e)$ for every $e \in E' \cup \bigcup_{j=i+1}^{r} F_j$.

The theorem now follows since H is even.

Cycle double cover

Definition: A cycle double cover of a graph G is a collection of cycles of G such that every edge of G appears in exactly two of them.

Examples: Eulerian graphs and bridgeless planar graphs.

Conjecture: (Szekeres 1973) Every bridgeless graph has a cycle double cover.

Proposition A graph G admits a 4-NZF iff it has a CDC that forms three even subgraphs.

Proof: If G admits a 4-NZF, then it is the union of two even subgraphs G_1, G_2. The cycles of G_1, G_2 and $G_1 \Delta G_2$ yield the desired cover. Let H_1, H_2 and H_3 be even graphs formed by the cover, then $G = H_1 \cup H_2$. Hence, G admits a 4-NZF.