Graphs & Algorithms: Advanced Topics
Let $G$ be a graph. A set $S \subseteq V(G)$ is called an $(f(n), \alpha)$-separator if

- $|S| \leq f(|V(G)|)$ and
- components of $G - S$ are of order $\leq \alpha n$.

**Theorem** Every tree contains a $(1, \frac{1}{2})$-separator, which can be found in $O(n)$ time.
Divide-and-Conquer method

1. Solve the problem on “very-very small” sets with brute force.

2. Otherwise **DIVIDE**: Find a “very small” vertex set $C$ “fast” such that $G - C$ falls into two “small” pieces $A$ and $B$ with no edges in between.

3. **CONQUER**: Explore all solutions restricted to $C$ (brute force) and solve the corresponding subproblems on $A$ and $B$ recursively. Put together the partial solutions.

Here:

“Small” means $< \beta n$, where $\beta < 1$ is a constant.

“very small” means $o(n)$.

“very-very small” means constant.

Outcome: Algorithm with subexponential running time $2^{\text{very small}}$
Separator for planar graphs

**Theorem** (Lipton-Tarjan, 1979) $G$ is planar with $n$ vertices. Then $G$ has a $(\sqrt{8n}, \frac{2}{3})$-separator, which can be found in $O(n)$-time.

**Remark** The order $\sqrt{n}$ is best possible for the order $f(n)$ of a separator $(f(n), \alpha)$ with constant $\alpha < 1$.

**Remark** Sparsity alone is not enough for the existence of a good separator. Most graphs of linear size would not allow a good separator with $f(n) = o(n)$.
A spanning tree could help

**Lemma** Let $G$ be a planar graph, and let $T$ be a spanning tree of $G$ with diameter $s$. Then a $(s + 1, \frac{2}{3})$-separator of $G$ can be found in $O(n)$-time.

*Proof of Theorem using Lemma*

WLOG $G$ is connected.

Fix $v_0 \in V(G)$ arbitrarily.

Define levels: $L_i := \{v \in V(G) : \text{dist}(v, v_0) = i\}$.

Let $l := \max\{i : L_i \neq \emptyset\}$.

Let $s := \lceil \sqrt{\frac{n}{2}} \rceil$ and $S_j := \bigcup\{L_i : i \equiv j \pmod{s}\}$.

Remove $S_{j_0}$ with $|S_{j_0}| \leq \lceil \frac{n}{s} \rceil \approx \sqrt{2n}$.

**Case 1.** All components of $G - S_{j_0}$ are of order $\leq \frac{2}{3}n$.

Be happy, you are done.
Case 2. There is one component $K, |K| > \frac{2}{3}n$.

$$K \subseteq \bigcup_{i=j+1}^{j+s-1} L_i$$ for some $j \equiv j_0 \pmod{s}$.

Then contract $L_j$ into a vertex in $G[K \cup L_j]$.

The resulting graph $H$ has a spanning tree with diameter $2(s - 1)$ so by Lemma we have a $(2s - 1, \frac{2}{3})$-separator $S_H$ in $H$.

Then $S_{j_0} \cup S_H$ is the appropriate separator of $G$. 
Proof of Lemma

Lemma $G$ is a planar graph, $T$ is a spanning tree of $G$ with diameter $s$. Then a $(s + 1, \frac{2}{3})$-separator of $G$ can be found in $O(n)$-time.

Proof. WLOG $G$ is a triangulation. (linear time!)

For $e \in E(G) \setminus E(T)$ there is a unique cycle $C(e)$ in $T + e$.

$n_{\text{int}}(C(e))$ is the number of vertices in $\text{Int}(C(e))$.

$n_{\text{ext}}(C(e))$ is the number of vertices in $\text{Ext}(C(e))$.

We are looking for an edge $e$ such that both $n_{\text{int}}(C(e))$ and $n_{\text{ext}}(C(e))$ are $\leq \frac{2}{3}n$. 
Proof of Lemma cont’d

Let \( e = xy \in E \setminus E(T) \) be arbitrary

Suppose \( n_{\text{int}}(C(e)) > \frac{2}{3} n \).

Find \( e' \in E \setminus E(T) \), s. t.

- \( n_{\text{ext}}(C(e')) \leq \frac{2}{3} n \).
- \( \text{Int}(C(e')) \subset \text{Int}(C(e)) \)

Let \( z \in \text{Int}(C(e)) \) be the third vertex in the face \( F \) containing \( e \).

**Case 1.** \( zx \in E(T) \).
Choose \( e' = zy \). (Note that \( zy \notin E(T) \).)

**Case 2.** \( zx, zy \notin E(T) \).
Choose \( e' = zx \) if \( n_{\text{int}}(C(zx)) \geq n_{\text{int}}(C(zy)) \)
Otherwise choose \( e' = zy \).
The Algorithm

**Input:** Plane triangulation $G$, spanning tree $T \subseteq G$

**Output:** Edge $e \in E \setminus E(T)$; $C(e)$ is a separator with $n_{ext}(C(e)), n_{int}(C(e)) \leq \frac{2}{3}n$.

$e = xy \in E \setminus E(T)$ arbitrary, with direction. Run Clockwise-DFS($y, x, x$) to determine $n_{int}(C(e))$ and $n_{int}(C(e))$.

IF $n_{ext}(C(e)) > \frac{2}{3}n$ THEN

Update $y := x, x := y$ (e changes direction)

IF $n_{int}(C(e)) > \frac{2}{3}n$.

WHILE $n_{int}(C(e)) > \frac{2}{3}n$ DO

$z \in C(e) \cup \text{Int}(C(e))$ such that \{z, x, y\} is a face.

Alternately run Clockwise-DFS($z, y, x$) and Anticlockwise-DFS($z, x, y$).

IF Clockwise-DFS terminates first THEN

$n_{int}(C(zx)) \leq n_{int}(C(zy))$; Update $e := zy$.

ELSE

Update $e := zx$.

Output $e$. 

8
Algorithm Clockwise-DFS

Clockwise-DFS(z, y, x)

Input: plane triangulation $G$, spanning tree $T \subseteq G$, cyclic lists $L_v$ of the neighbors of $v \in V$ in $G$, those which are also neighbors in $T$ are marked; root vertex $z$, reference vertex $y$, target vertex $x$.

$u := y, v := z$.

WHILE $v \neq x$ DO

$u := v$,

$v := T$-neighbor of $v$ coming first after $u$ in $L_v$ according to the anticlockwise direction.

Remark The tree produced by Clockwise-DFS tends to “bend” in the clockwise direction.

For Anticlockwise-DFS: Replace “clockwise” with “anticlockwise”.


Finding Planar independent sets

MAXIMUM (PLANAR) INDEPENDENT SET PROBLEM

Input: (Planar) graph $G$
Output: Independent set $X \subseteq V$ with maximum cardinality, that is, $|X| = \alpha(G)$.

**Theorem** The MAXIMUM PLANAR INDEPENDENT SET problem is NP-hard.

**Theorem** The MAXIMUM PLANAR INDEPENDENT SET problem can be solved in time $2^{O(\sqrt{n})}$.

**Remark** We don’t know whether it is possible to solve the MAXIMUM INDEPENDENT SET problem in time $2^{o(n)}$. In fact, we don’t expect that happening.
Algorithm $\text{PlanarIndSet}$

Input: Plane graph $G$
Output: Maximum independent set $I$

IF $|V(G)| \leq 1$ THEN
    $I := V(G)$
ELSE
    $I := \emptyset$
    Find a $(\sqrt{8}|V(G)|, \frac{2}{3})$-separator $C$ for $G$.
    Let $A \cup B = V \setminus C$ a partition of $V$ such that
    $|A|, |B| \leq \frac{2}{3}n$, $E(A, B) = \emptyset$.
    FOR ALL independent set $S \subseteq C$ DO
        $I_A := \text{PlanarIndSet}(G[A \setminus N(S)])$
        $I_B := \text{PlanarIndSet}(G[B \setminus N(S)])$
        IF $|S| + |I_A| + |I_B| > |I|$ THEN
            $I := S \cup I_A \cup I_B$
    output $I$