Dynamic Programming on Trees

Example: Independent Set on $T = (V, E)$ rooted at $r \in V$.

For $v \in V$ let T_v denote the subtree rooted at v.

Let $f^+(v)$ be the size of a maximum independent set for T_v that contains v. Similarly, $f^-(v)$ is the size of a maximum independent set for T_v that does not contain v.

The following algorithm computes a maximum independent set for T in $O(|V|)$ time.

Traverse T starting from r in post-order. Let v be the current vertex.

- If v is a leaf, let $f^+(v) = 1$ and $f^-(v) = 0$.
- Else let $x_1, \ldots, x_k$ be the children of v. Set $f^+(v) = 1 + \sum_{i=1}^{k} f^-(x_i)$ and $f^-(v) = \sum_{i=1}^{k} \max\{f^+(x_i), f^-(x_i)\}$.

Return $\max\{f^+(r), f^-(r)\}$.

Tree Decompositions

Definition. A tree decomposition for a graph $G = (V, E)$ is a pair

$$(\,\{X_i \,|\, i \in I\},\ \ T = (I, F)\,)$$
$$\text{bags} \qquad\qquad \text{tree}$$

such that

- $\bigcup_{i \in I} X_i = V$ (bags cover vertices);
- for each $\{u, v\} \in E$ there is some $i \in I$ s.t. $\{u, v\} \subseteq X_i$ (bags cover edges);
- for all $v \in V$ the set $I_v = \{i \in I \,|\, v \in X_i\}$ is connected in T (tree property).

The **width** of a tree decomposition is

$$\max_{i \in I} |X_i| - 1.$$

The **treewidth** of a graph is the minimum width of a tree decomposition for it.

Example. Trees have treewidth 1.

Basic Observations

Observation. For any graph $G = (V, E)$ a single bag containing V forms a tree decomposition of width $n-1$.

We are interested in tree decompositions of small width, which certify that the graph is in some way "tree-like".

Denote the treewidth of a graph G by $\text{tw}(G)$.

Proposition. $\text{tw}(H) \leq \text{tw}(G)$ for any subgraph H of a graph G.

Proposition. If a graph $G = (V, E)$ has two components A and B with $A \cup B = V$ then $\text{tw}(G) = \max\{\text{tw}(A), \text{tw}(B)\}$.

Treewidth of cliques and grids

Lemma. Let $(\{X_i \,|\, i \in I\}, T = (I, F))$ be a tree decomposition for $G = (V, E)$. For any clique $G[W]$, $W \subseteq V$, there is an $i \in I$ such that $W \subseteq X_i$.

Proof. Root T arbitrarily. For $w \in W$ let r_w denote the bag containing w with minimum height. Then the bag from $\{r_w \,|\, w \in W\}$ with maximum height contains W. □

In particular, the treewidth of K_n is $n-1$.

The $n \times n$-grid on $\{(i,j) \,|\, 1 \leq i,j \leq n\}$ has treewidth $\leq n$: Consider the path on

$$\begin{aligned} X_{n(i-1)+j} &= \{(i,k) \,|\, j \leq k \leq n\} \cup \\ &\quad \{(i+1,k) \,|\, 1 \leq k \leq j\}, \end{aligned}$$

$1 \leq i \leq n-1$, $1 \leq j \leq n$.

How many vertices are needed in T?

Definition. A tree decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ of width k is **smooth** if

- $|X_i| = k + 1$ for all $i \in I$;
- $|X_i \cap X_j| = k$ for all $\{i, j\} \in F$.

Proposition. For any graph with treewidth k there exists a smooth tree decomposition of width k. $\rightarrow$ Exercise

Lemma. If $(X, T = (I, F))$ is a smooth tree decomposition of width k for $G = (V, E)$ then $|I| = |V| - k$. $\rightarrow$ Exercise

In particular, $n(T) \leq n(G)$.

Number of edges

Lemma. A graph $G = (V, E)$ of treewidth at most k has at most $k|V| - \binom{k+1}{2}$ edges.

Proof. Induction on $|V|$. Base case is $|V| = k+1$. Consider a smooth tree decomposition $(\{X_i \,|\, i \in I\}, T = (I, F))$ for G and a leaf i of T. Then there is a unique vertex $v \in X_i$ that does not belong to any other X_j, $j \neq i$. Clearly $\deg_G(v) \leq k$. Removing i from T yields a tree decomposition for $G[V \setminus \{v\}]$. $\square$

Corollary. A graph has treewidth at most one if and only if it is a forest.

Treewidth and Cuts

Lemma. Let $(\{X_i \mid i \in I\}, T = (I, F))$ be a tree decomposition for a connected graph $G = (V, E)$ such that $X_i \not\subseteq X_j$ for all $i, j \in I$. Then

a) $X_i \cap X_j$ is a cut in G for any $\{i, j\} \in F$;

b) X_i is a cut in G for any $i \in I$ that is not a leaf in T.

Remark. It is possible to adapt any tree decomposition in $O(|I|)$ time to fulfill the non-containment condition without changing its width.

Treewidth and Separators

Theorem. From a given tree decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ of width k for a graph $G = (V, E)$ one can find a $(k+1, \frac{1}{2})$-separator for G in $O(|I|)$ time.

Proof. Root T arbitrarily and define a weight function w on I by $w(i) = |X_i \setminus X_{\text{parent}(i)}|$. Each $v \in V$ is counted exactly once (bags containing v are connected).

Therefore, $\sum_{i \in I} w(i) = |V|$. By the Separator Theorem for (weighted) trees we obtain a $(1, \frac{1}{2})$-separator s for T.

Removing X_s disconnects G where

- any $v \in V \setminus X_s$ can appear in at most one subtree (otw, it would also appear in X_s by connectivity);
- each subtree defines at least one component (no edge between subtrees);
- each subtree (and hence component) consists of at most $\frac{n}{2}$ vertices. $\square$

Dynamic Programming on Graphs of Treewidth at most k

Given: $G = (V, E)$ and smooth tree decomp. $(\{X_i \mid i \in I\}, T = (I, F))$ of width $\leq k$ for G. The algorithm below computes a maximum independent set for G in $O(k^2 4^k |V|)$ time.

Pick an arbitrary root $r \in I$ and for $i \in I$ let

$$V_i = \bigcup_{j \in T_i} X_j$$

where T_i denotes the subtree rooted at i.

For $U \subseteq X_i$ let $f^U(i)$ be the size of a maximum independent subset of V_i whose intersection with X_i is exactly U.

- Traverse T starting from r top-down. Let i be the current vertex.
- If i is a leaf, for every $U \subseteq X_i$ let $f^U(i) = |U|$, if U is independent in G and $f^U(i) = -\infty$, otherwise.
- Else let $c_1, \ldots, c_\ell$ be the children of i. Set

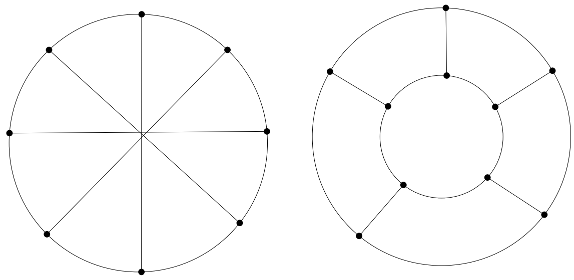
$$f^U(i) = |U| + \sum_{j=1}^{\ell} \max \Big\{ f^W(c_j) - |U \cap W| \;\Big|\; W \subseteq X_{c_j} \wedge W \cap X_i = U \cap X_{c_j} \wedge W \cup U \text{ independent} \Big\}$$

Closure Properties

Proposition. Graphs of treewidth at most k are closed under taking minors.

Proof. Removal of edges and isolated vertices are trivial. When contracting an edge $\{u, v\}$, replace all occurrences of u in any bag by v. $\square$

By Robertson-Seymour there is hence a finite set of forbidden minors. But they are not known, except for small k.

- $k = 0$: K_2.
- $k = 1$: K_3.
- $k = 2$: K_4.
- $k = 3$: K_5, $K_{2,2,2}$, 
- $k = 4$: more than 75...

Computing treewidth

Theorem [Arnborg, Corneil, Proskurowski '87]. Deciding whether the treewidth of a given graph is at most k is NP-complete.

Theorem [Bodlaender '96]. For any $k \in \mathbb{N}$ there exists a linear time algorithm to test whether a given graph has treewidth at most k and—if so—output a corresponding tree decomposition.

(The runtime is exponential in k^3.)

Open. Can the treewidth be computed in polynomial time for planar graphs?

Not everything is easy for bounded treewidth...

Theorem **[Nishizeki,Vygen,Zhou '01].** Edge-disjoint paths is NP-complete for graphs of treewidth 2.

(But trivial for trees and polynomial for outerplanar graphs.)

Given a graph $G = (V, E)$ and $\{s_i, t_i\} \in \binom{V}{2}$, $1 \leq i \leq k$, find k edge disjoint paths P_i such that P_i connects s_i and t_i.

Theorem **[McDiarmid/Reed '01].** Weighted coloring is NP-hard for graphs of treewidth 3.

(But trivial for bipartite graphs $\rightarrow$ trees.)

Given a graph $G = (V, E)$ and $w : E \rightarrow \mathbb{N}$, a weighted k-coloring is a function $c : V \rightarrow [k]$ such that $|c(u) - c(v)| \geq w(e)$ for all $\{u, v\} \in E$.

Cops and robber

In the *omniscient cops and robber game*, k cops each occupy a vertex of a graph in which a robber moves trying to escape capture. The robber moves along edges "at infinite speed", the cops move by helicopter.

Definition. Given a graph $G = (V, E)$ and $k \in \mathbb{N}$, a position in the k cops and robber game on G is a pair (C, r), where $C \in \binom{V}{k}$ (location of cops) and r is a vertex in some component of $G \setminus C$ (location of robber).

In Round 0, the cops choose $C_0 \in \binom{V}{k}$ and then the robber chooses $r_0 \in V \setminus C_0$ arbitrarily.

In Round i, $i > 0$, the cops choose $C_i \in \binom{V}{k}$ and then the robber chooses a vertex $r_i \in V \setminus C_i$ such that there is a path between r_i and r_{i-1} in $G \setminus (C_i \cap C_{i-1})$.

The cops win if after some finite number of rounds the robber has no vertex to choose.

Cops, robber, and treewidth

Theorem [Seymour/Thomas '93]. If a graph G has treewidth at most k then $k+1$ omniscient cops can catch a robber on G.

Proof. Suppose $n(G) > k+1$ and let $(\{X_i \mid i \in I\}, T = (I, F))$ be a smooth tree decomposition of width $\leq k$ for G.

Pick an arbitrary root $a \in I$ and for $i \in I$ let

$$V_i = \bigcup_{j \in T_i} X_j$$

where T_i denotes the subtree rooted at i.

In the first round choose $C_0 = X_a$.

In Round j, we suppose $C_{j-1} = X_b$ for some $b \in I$ and $r_{j-1} \in V_b \backslash X_b$. Let c be the child of b for which $r_{j-1} \in V_c$. Observe that $X_b \cap X_c$ is a k-cut in G. Thus choosing $C_j = X_c$ confines the robber to $V_c \setminus X_c$.

After a finite number of steps the game will arrive at a leaf ℓ of T for which $V_\ell \setminus X_\ell = \emptyset$ and the robber has nowhere to go. $\square$

Remark. The converse also holds but the proof is much more involved.

Cops and robber on the grid

Proposition. On the $n \times n$-grid $n-1$ omniscient cops cannot catch a robber.

Proof. Whichever positions the $n-1$ cops choose to occupy, there is always at least one row and at least one column of the grid without any cop.

We show that the robber can always move to the intersection of a cop-free row with a cop-free column.

Initially, this is clear.

Suppose that at some point one or more cops enter the free row and/or free column where the robber is located. Then the robber can move along the previously free row to the to-be free column and within this column to the to-be free row. □

Proposition. On the $n \times n$-grid n omniscient cops cannot catch a robber, for $n \geq 2$.
→ Exercise.

Corollary. The $n \times n$-grid has treewidth n.

Corollary. There are planar graphs on n vertices whose treewidth is $\Omega(\sqrt{n})$.

Partial k-trees

Definition. A k-**tree** is a graph formed from a k-clique by iteratively joining a new vertex to some k-clique.
In other words, a graph is a k-tree $\iff$ there is an order $\pi = (v_1, v_2, \ldots, v_n)$ of its vertices such that the neighbors of v_i preceding it in π form a $\min\{i-1, k\}$-clique, for all $1 \leq i \leq n$.

Observation.

a) 1-trees are exactly trees.

b) A k-tree on $n \geq k$ vertices has $kn - \binom{k+1}{2}$ edges.

Definition. A graph is a **partial** k-**tree** if it is a subgraph of a k-tree.

Partial k-trees and treewidth

Theorem. A graph G is partial k-tree $\iff$ G has treewidth at most k.

Proof. "$\Leftarrow$": Let $(\{X_i \mid i \in I\}, T = (I, F))$ be a smooth tree decomposition of width $\leq k$ for G. Add all edges inside bags to G.
Claim. Resulting graph H is a k-tree.
Induction on $|I|$:

$|I| = 1$: H is K_{k+1}, a k-tree.

Otherwise, let $i \in I$ be a leaf of T. Then there is a $v \in X_i$ that does not occur in any X_j, $j \in I \setminus \{i\}$. Removal of i from I results in a tree decomposition of width $\leq k$ for $H' = H - v$.

By induction H' is a k-tree. Adding v to H' and connect it to the k-clique $X_i \setminus \{v\}$, we get H, which hence is a k-tree. $\square$

Partial k-trees and treewidth

Theorem. A graph G is partial k-tree $\iff$ G has treewidth at most k.

Proof. "$\Rightarrow$": Let H be a k-tree containing G and $\pi = (v_1, \ldots, v_n)$ a vertex order for H such that the neighbors of v_i preceding it in π form a $\min\{i-1, k\}$-clique, for all $1 \leq i \leq n$.

Build a tree decomposition of width k for $V_i = \{v_1, \ldots, v_i\}$ inductively such that for every j, $1 \leq j \leq i$, there is a bag that contains $\{v_j\} \cup \Pi_j$, where $\Pi_j = V_j \cap N_H(v_j)$.

$i \leq k+1$: A single bag for V_i suffices.

Otw, let $\ell = \max\{1 \leq \ell < i \mid v_\ell \in \Pi_j\}$. By the induction hypothesis there is a tree decomposition of width k for V_{i-1} in which one bag X_a contains $\{v_\ell\} \cup \Pi_\ell$.

Create a new node b, make it adjacent to a only, and set $X_b = \{v_i\} \cup \Pi_i$. (Note that $\Pi_i \subseteq \{v_\ell\} \cup \Pi_\ell$ because Π_i is a clique.) $\square$

Grids, minors, and treewidth

Theorem [Alon, Seymour, Thomas '90].
For any fixed graph H, every graph G that does not contain H as a minor has treewidth at most $n(H)^{3/2}\sqrt{n(G)}$.

Corollary. A planar graph on n vertices has treewidth $O(\sqrt{n})$.

Theorem [Robertson, Seymour, Thomas '94].
Every graph of treewidth larger than 20^{2k^5} has a $k \times k$-grid as a minor.
There are graphs of treewidth $\Omega(k^2 \log k)$ that do not have a $k \times k$-grid as a minor.