

Graphs & Algorithms: Advanced Topics

Basic concepts, definitions and facts

Things

Prerequisite: basic graph theory and graph algorithms. In particular the material of the course Graphs and Algorithms (Spring 2008)

In graph theoretic notation we mostly follow the book “Introduction to Graph Theory” by Doug West.

Graphs – Definition

A **graph** G is a pair consisting of

- a **vertex** set $V(G)$, and
- an **edge** set $E(G) \subseteq \binom{V(G)}{2}$.

If there is no confusion about the underlying graph we often just write $V = V(G)$ and $E = E(G)$.

x and y are the **endpoints** of edge $e = \{x, y\}$.

They are called **adjacent** or **neighbors**.

e is called **incident** with x and y .

A **loop** is an edge whose endpoints are equal.

Multiple edges have the same set of endpoints. In the definition of a “graph” we don’t allow loops and multiple edges. To emphasize this, we often say “simple graph”. When we do want to allow multiple edges or loops, we say **multigraph**.

Remarks A multigraph might have no multiple edges or loops. Every (simple) graph is a multigraph, but not every multigraph is a (simple) graph.

Special graphs_____

K_n is the complete graph on n vertices.

$K_{n,m}$ is the complete bipartite graph with partite sets of sizes n and m .

P_n is the path on n vertices

C_n is the cycle on n vertices

Further definitions and notation_____

The **degree** of vertex v is the number of edges incident with v . Loops are counted twice.

A set of pairwise adjacent vertices in a graph is called a **clique**. A set of pairwise non-adjacent vertices in a graph is called an **independent set**.

A graph G is **bipartite** if $V(G)$ is the union of two (possibly empty) independent sets of G . These two sets are called the **partite sets** of G .

The **complement** \overline{G} of a graph G is a graph with

- vertex set $V(\overline{G}) = V(G)$ and
- edge set $E(\overline{G}) = \binom{V}{2} \setminus E(G)$.

H is a **subgraph** of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. We write $H \subseteq G$. We also say G **contains** H and write $G \supseteq H$.

For a subset $S \subseteq V(G)$ define $G[S]$, the **induced subgraph** of G on S : $V(G[S]) = S$ and $E(G[S]) = \{e \in E(G) : \text{both endpoints are in } S\}$.

Leaves, trees, forests..._____

A graph with no cycle is **acyclic**. An acyclic graph is called a **forest**.

A connected acyclic graph is a **tree**.

A **leaf** (or **pendant vertex**) is a vertex of degree 1.

A **spanning subgraph** of G is a subgraph with vertex set $V(G)$.

A **spanning tree** is a spanning subgraph which is a tree.

Examples. Paths, stars

Properties of trees

Lemma. T is a tree, $n(T) \geq 2 \Rightarrow T$ contains at least two leaves.

Deleting a leaf from a tree produces a tree.

Theorem (Characterization of trees) For an n -vertex graph G , the following are equivalent

1. G is connected and has no cycles.
2. G is connected and has $n - 1$ edges.
3. G has $n - 1$ edges and no cycles.
4. For each $u, v \in V(G)$, G has exactly one u, v -path.

Corollary.

- (i) Every edge of a tree is a cut-edge.
- (ii) Adding one edge to a tree forms exactly one cycle.
- (iii) Every connected graph contains a spanning tree.

Walks, trails, paths, and cycles_____

A **walk** is an alternating list $v_0, e_1, v_1, e_2, \dots, e_k, v_k$ of vertices and edges such that for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i .

A **trail** is a walk with no repeated edge.

A **path** is a walk with no repeated vertex.

A u, v -walk, u, v -trail, u, v -path is a walk, trail, path, respectively, with first vertex u and last vertex v .

If $u = v$ then the u, v -walk and u, v -trail is **closed**. A closed trail (without specifying the first vertex) is a **circuit**. A circuit with no repeated vertex is called a **cycle**.

The **length** of a walk trail, path or cycle is its number of edges.

Connectivity

G is **connected**, if there is a u, v -path for every pair $u, v \in V(G)$ of vertices.

Otherwise G is **disconnected**.

Vertex u is **connected to** vertex v in G if there is a u, v -path. The **connection relation** on $V(G)$ consists of the ordered pairs (u, v) such that u is connected to v .

Claim. The connection relation is an equivalence relation.

Lemma. Every u, v -walk contains a u, v -path.

The **connected components** of G are its maximal connected subgraphs (i.e. the equivalence classes of the connection relation).

An **isolated vertex** is a vertex of degree 0. It is a connected component on its own, called **trivial** connected component.

Planar graphs — a recap_____

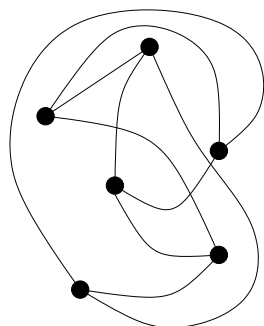
A **drawing** of a multigraph G is a function f defined on $V(G) \cup E(G)$ that assigns

- a point $f(v) \in \mathbb{R}^2$ to each vertex v and
- an $f(u), f(v)$ -curve to each edge uv ,

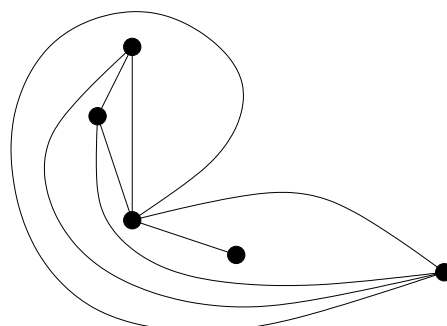
such that the images of vertices are distinct. A point in $f(e) \cap f(e')$ that is not a common endpoint is a **crossing**.

A graph is **planar** if it has a drawing without crossings. Such a drawing is a **planar embedding** of G . A planar (multi)graph *together* with a particular planar embedding is called a **plane (multi)graph**.

drawing



plane embedding



Jordan curves

A **curve** is a subset of \mathbb{R}^2 of the form

$$\alpha = \{\gamma(x) : x \in [0, 1]\} ,$$

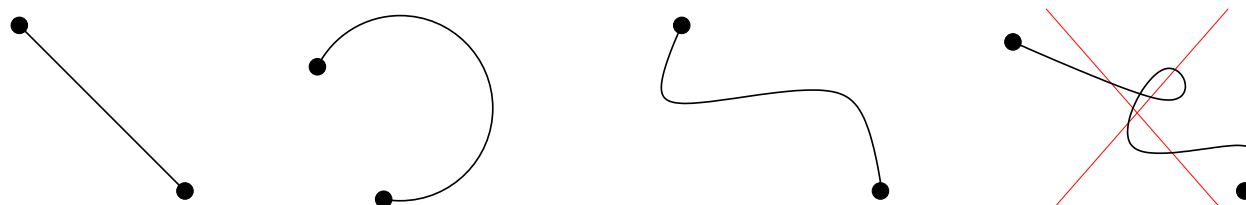
where $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ is a continuous mapping from the closed interval $[0, 1]$ to the plane. $\gamma(0)$ and $\gamma(1)$ are called the *endpoints* of curve α .

A curve is **closed** if its first and last points are the same. A curve is **simple** if it has no repeated points except possibly first = last. A closed simple curve is called a **Jordan-curve**.

Examples: Line segments between $p, q \in \mathbb{R}^2$

$$x \mapsto xp + (1 - x)q ,$$

circular arcs, Bezier-curves without self-intersection, etc...

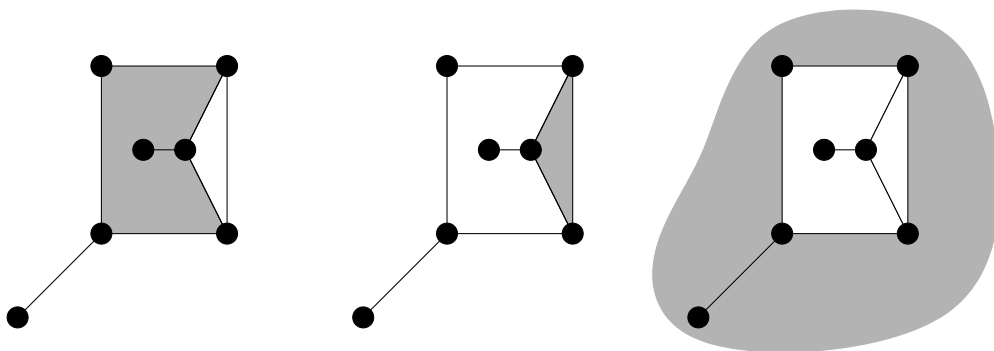


Regions and faces

An **open set** in the plane is a set $U \subseteq \mathbb{R}^2$ such that for every $p \in U$, all points within some small distance belong to U . A **region** is an open set U that contains a u, v -curve for every pair $u, v \in U$.

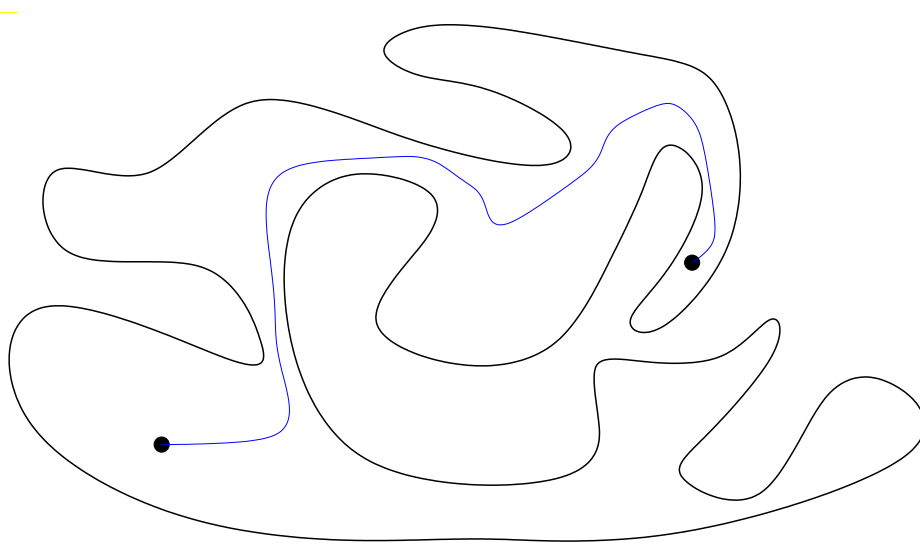
The **faces** of a plane multigraph are the maximal regions of the plane that contain no points used in the embedding.

A finite plane multigraph G has one **unbounded face** (also called **outer face**).

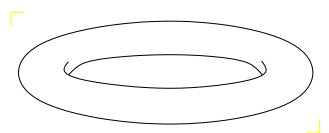


The unconscious ingredient.

Jordan Curve Theorem. A simple closed curve C partitions the plane into exactly two faces, denoted by $\text{Int}(C)$ and $\text{Ext}(C)$, each having C as boundary.



Not true on the torus!





Euler's Formula

Theorem. (Euler, 1758) If a plane multigraph G with k components has n vertices, e edges, and f faces, then

$$n - e + f = 1 + k.$$

Applications If G is a simple, planar graph with $n(G) \geq 3$, then

$$e(G) \leq 3n(G) - 6.$$

If G is a simple plane graph with $n(G) \geq 3$ vertices then

$$f(G) \leq 2n(G) - 4.$$