Graphs & Algorithms II  Exercise Set 11  HS08

Exercise 32

a) Show that for any set P of n points in $\mathbb{R}^2$ there are at most $O(n^{3/2})$ pairs of points in $\binom{P}{2}$ that have Euclidean distance exactly one. (*Hint: Show that the unit distance graph does not contain $K_{2,3}$ as a subgraph.*)

b) Show that for any set P of n points in $\mathbb{R}^3$ there are at most $O(n^{5/3})$ pairs of points in $\binom{P}{2}$ that have Euclidean distance exactly one.

Exercise 33

a) Let $G = (V,E)$ be a graph with average vertex degree at least $2k$. Show that there exists a subgraph $H$ in $G$ in which every vertex has degree at least $k$.

b) Let $T$ be a tree on $k$ vertices. Prove that $\frac{n(k-2)}{2} \leq \text{ex}(n,T) < nk$, for any $n \in \mathbb{N}$ such that $n \mod (k-1) \equiv 0$.

*Remark:* The lower bound is conjectured to be tight.

Exercise 34

Consider the finite field $\mathcal{F}_p$ for some prime number $p$ and define a graph $G_p = (V_p,E_p)$ on the vertex set $V_p = \mathcal{F}_p^2 \setminus \{(0,0)\}$. Two points $(x,y)$ and $(x',y')$ from $V_p$ are connected by an edge in $G_p$ if and only if they are distinct and $xx' + yy' \equiv 1 \mod p$.

a) Prove that $G_p$ does not contain $K_{2,2}$ as a subgraph.

b) Show that $|E_p| \geq \frac{1}{4}(p-1)(p^2-1)$.

c) Conclude that $\text{ex}(n,K_{2,2}) = \Omega(n^{3/2})$.

*Remark:* Together with Homework 11 it follows that $\text{ex}(n,K_{2,2}) = \Theta(n^{3/2})$.

Homework 11

a) Let $G = (V,E)$ be a graph which does not contain $K_{s,t}$ as a subgraph, for any $s,t \in \mathbb{N}$. Prove that $\sum_{v \in V} \binom{d(v)}{s} \leq (t-1)\binom{n}{s}$.

b) Prove that for any $s,t \in \mathbb{N}$ there is some constant $c \in \mathbb{R}$ such that $\text{ex}(n,K_{s,t}) \leq cn^{2-1/s}$, for any $n \in \mathbb{N}$.

*Homework due:* 4.12.2008, 11:00AM.