Homework 9

Show that for every graph \( G = (V, E) \) there is an equitable \( (\Delta(G) + 1) \)-coloring of its edges.

Exercise 27

Brook's Theorem says that any connected graph \( G \) that is not a clique nor an odd cycle can be colored with at most \( \Delta(G) \) colors.

Find the mistake in the following “proof” of Brook’s Theorem.

Induction on \( n = |V| \). For \( n \leq 2 \) the only connected graph is a clique and there is nothing to show. For the induction step take a minimum size vertex cut \( S \subseteq V \) (a cut exists because \( G \) is not a clique). Clearly \( |S| \leq \Delta(G) \). Let \( H_1, \ldots, H_k \) be the \( k \geq 2 \) -jobs. By the induction hypothesis every \( H_i \) is \( \Delta(G) \)-colorable. Permute the labels of the colors such that they agree on the \( \leq \Delta(G) \) vertices of \( S \) to obtain a proper \( \Delta(G) \)-coloring of \( G \).

Exercise 28

Let \( G = (V, E) \) be a bipartite graph with color classes \( V_1 \) and \( V_2 \) and \( |V_1| = |V_2| = k \). Prove: If each vertex \( v \in V \) has at least \( \frac{k}{2} \) neighbors then \( G \) contains a perfect matching.

Exercise 29

Let \( G = (V, E) \) be a tripartite graph with color classes \( V_1, V_2, \) and \( V_3 \) such that \( |V_1| = |V_2| = |V_3| = k \). Prove: If each vertex \( v \in V \) has at least \( \frac{2k}{3} \) neighbors in each of the other two classes then \( G \) contains at least \( k - 2 \) pairwise vertex-disjoint triangles.

*Hint*: Partition \( V \) into triples consisting of one vertex from each color class. Show that...

a) One may suppose that in every triple two chosen vertices are adjacent in \( G \).

b) For any three triples whose vertices do not induce a triangle in \( G \) one can find a fourth triple that together with the three induces two triangles.

Homework due: 20.11.2008, 11:00 AM.