Planar graphs — a recap

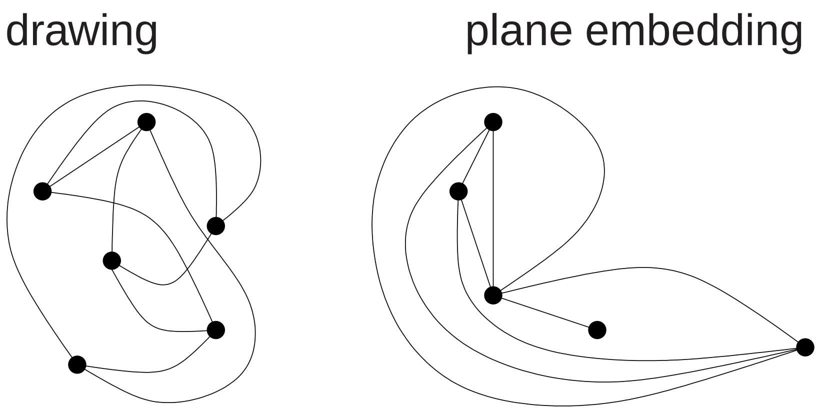
A drawing of a multigraph G is a function f defined on $V(G) \cup E(G)$ that assigns

- a point $f(v) \in I\!\!R^2$ to each vertex v and
- a curve $f(e)$ with endpoints $f(u)$ and $f(v)$ to each edge $e = uv$,

such that the images of vertices are distinct, and if $f(u) \in f(e)$ then $f(u)$ is an endpoint of $f(e)$.

A point in $f(e) \cap f(e')$ that is not a common endpoint is a crossing.

A graph is planar if it has a drawing without crossings. Such a drawing is a planar embedding of G. A planar (multi)graph *together* with a particular planar embedding is called a plane (multi)graph.



Jordan curves

A curve is a subset of $\mathbb{R}^2$ of the form

$$\alpha = \{\gamma(x) : x \in [0,1]\} ,$$

where $\gamma : [0,1] \to \mathbb{R}^2$ is a continuous mapping from the closed interval $[0,1]$ to the plane. $\gamma(0)$ and $\gamma(1)$ are called the *endpoints* of curve α.

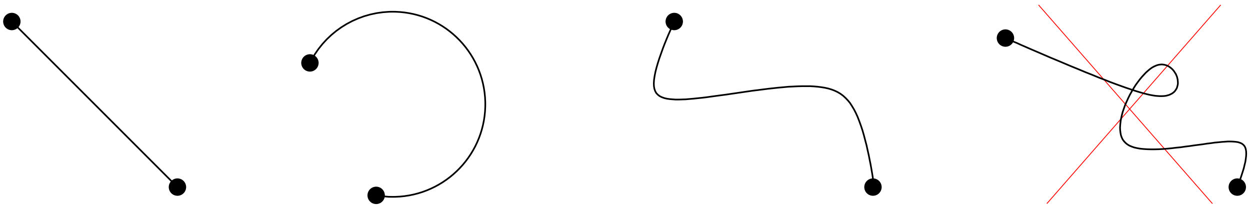
A curve is closed if its first and last points are the same.

A curve is simple if it has no repeated points except possibly first = last.

Examples. Line segments between $p, q \in \mathbb{R}^2$

$$x \mapsto xp + (1-x)q ,$$

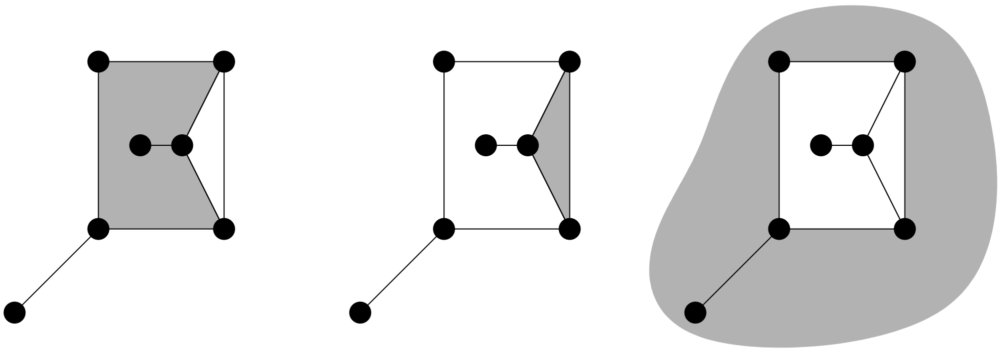
circular arcs, Bezier-curves without self-intersection, etc...



Regions and faces

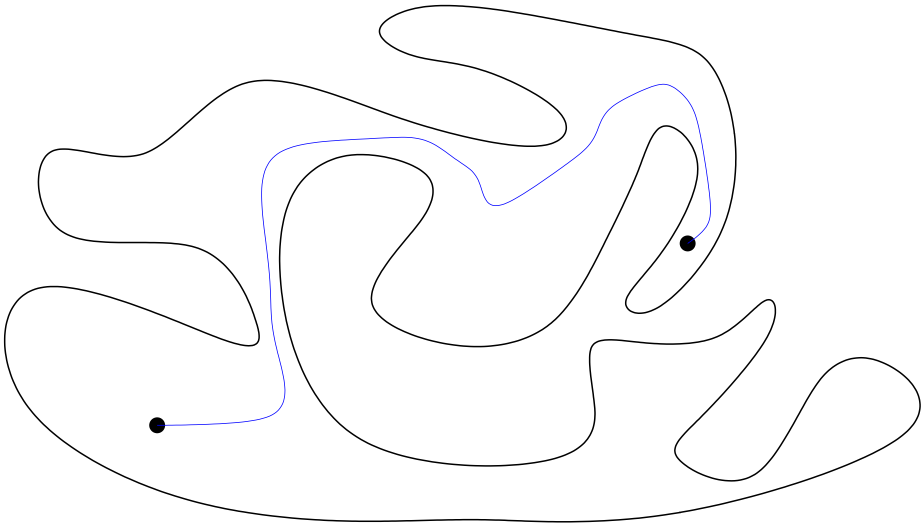
An open set in the plane is a set $U \subseteq R^2$ such that for every $p \in U$, all points within some small distance belong to U. A connected region is an open set U that contains a u, v-curve for every pair $u, v \in U$.

The faces of a plane multigraph are the maximal connected regions of the plane that contain no points used in the embedding.
A finite plane multigraph G has one unbounded face (also called outer face).

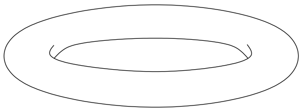


The unconscious ingredient

Jordan Curve Theorem. A simple closed curve C partitions the plane into exactly two faces, denoted by $\mathtt{Int}(C)$ and $\mathtt{Ext}(C)$, each having C as boundary.



Not true on the torus!



Euler's Formula

Theorem. (Euler, 1758) If a plane multigraph G with k components has n vertices, e edges, and f faces, then

$$n - e + f = 1 + k.$$

Corollary 1. If G is a simple, planar graph with $n(G) \geq 3$, then

$$e(G) \leq 3n(G) - 6.$$

If, additionally, G contains no triangles then

$$e(G) \leq 2n(G) - 4.$$

Corollary of corollary. K_5 and $K_{3,3}$ are non-planar.

Corollary 2. If G is a simple plane graph with $n(G) \geq 3$ vertices then

$$f(G) \leq 2n(G) - 4.$$

Kuratowski's Theorem

The subdivision of edge $e = xy$ is the replacement of e with a new vertex z and two new edges xz and zy. The graph H' is a subdivision of H, if one can obtain H' from H by a series of edge subdivisions. Vertices of H' with degree at least three are called branch vertices.

Theorem (Kuratowski, 1930). A graph G is planar iff G does not contain a subdivision of K_5 or $K_{3,3}$.

Proof by contradiction.
A Kuratowski subgraph of G is a subgraph of G that is a subdivision of K_5 or $K_{3,3}$.

Assume that G is a minimal counterexample to Kuratowski's Theorem, i.e., G is nonplanar and does not contain a Kuratowski subgraph, and every proper subgraph is planar (since a subgraph cannot contain a Kuratowski subgraph, either).

Obtain a contradtiction by the following lemmas.

The spine of the proof: θ-graphs

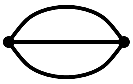
Lemma 1 (Deleting endpoints). Let G be a minimal counterexample to Kuratowski's theorem.

(a) G is simple, and every vertex has degree ≥ 3.

Moreover, for every edge $e = xy$ of G:

(b) In $G - x - y$, every vertex has degree ≥ 2.

(c) $G - x - y$ does not contain a θ-graph,

i.e., a subdivision of 

Lemma 2 (K_5 and $K_{3,3}$) For a simple graph K of minimum degree ≥ 3, the following are equivalent:

(i) For every edge xy of K, the graph $K - x - y$ has minimum degree ≥ 2 and does not contain a θ-graph.

(ii) For every edge xy of K, the graph $K - x - y$ is a cycle (on $n \geq 3$ vertices)

(iii) K is isomorphic to either K_5 or $K_{3,3}$.

Note. The implications (iii) ⇒ (ii) ⇒ (i) are clear.

Proof of Lemma 1, (a) and (b)

Lemma 1 (Deleting endpoints). Let G be a minimal counterexample to Kuratowski's theorem.

(a) G is simple, and every vertex has degree ≥ 3.

Moreover, for every edge $e = xy$ of G:

(b) In $G - x - y$, every vertex has degree ≥ 2.

Proof.

(a) Multiple edges, loops, and vertices of degree ≤ 1 do not affect planarity and cannot be used for Kuratowski subgraphs.

Thus, they can be deleted and hence do not occur in a minimal counterexample.

Vertices of degree 2 can be used to "un-subdivide" edges.

(b) Vertices in $G-x-y$ have degree ≥ 1, by (a). If p is a hanging vertex of $G - x - y$, then $px, py \in E(G)$. By minimality, $G - xy$ is planar. Take a drawing and draw the edge xy along px and py (possible since at most one additional edge emanates from p).

Proof of Lemma 1 (c)

Lemma 1 (Deleting endpoints). Let G be a minimal counterexample to Kuratowski's theorem.

(c) $G - x - y$ does not contain a θ-graph.

Exercise. If the contracted graph G/xy contains a subdivision of $K_{3,3}$ then so does G. If G/xy contains a subdivision of K_5 then G contains a subdivision of K_5 or of $K_{3,3}$.

Proof of Lemma 1 (c). By the exercise, G/xy is planar. Fix a plane drawing f of G/xy. Let B be the boundary of the face of f restricted to $G/xy - \{xy\} = G - x - y$ containing xy.

Note. The boundary of a face does not contain a θ.

Claim. $B = G - x - y$.

Else, $G - x - y - B$ contains $\geq$ 1 edge e, $f(e)$ must lie outside the face determined by B. Thus, B contains a cycle C such that xy lies inside C and e lies outside C. Let R be the set of all edges that lie outside $f(C)$ in f. ($G-x-y-B-R$ may be nonempty!)

By minimality of G, there is a planar drawing g of $G - R$. Since xy is an edge, x and y must lies on the same side of $g(C)$, and so do all edges emanating from them.

Since B contains no θ, every component of $B - C$ intersects C in at most one point. So we may modify g to move all these components (together with the corresponding edges of $G - x - y - B - R$) inside $g(C)$. But then $G - R$ lies inside $g(C)$ in g.

Finally, we can modify g by drawing the edges in R outside of $g(C)$, as in f. This yields a plane drawing of G, a contradiction.

Proof of Lemma 2

Lemma 2 (K_5 and $K_{3,3}$) For a simple graph K of minimum degree ≥ 3, the following are equivalent:

(i) For every edge xy of K, the graph $K - x - y$ has minimum degree ≥ 2 and does not contain a θ-graph.

(ii) For every edge xy of K, the graph $K - x - y$ is a cycle (on $n \geq 3$ vertices)

(iii) K is isomorphic to either K_5 or $K_{3,3}$.

Proof of (i) ⇒ (ii). By (i), $K-x-y$ is a tree of cycles ⇒ leaf cycle C having at most one vertex v in common with the remaining graph, and at least two more vertices p and q. By min deg ≥ 3, each of p and q is joined to x or y in K. Thus, C together with the edges of K joining x, y, p, q contains a θ. Hence, each edge uv of $K - x - y$ has an endpoint on C, so $K - x - y = C$.

Proof of (ii) ⇒ (iii). If $n = 3$ then for any two vertices u, v of the cycle $K - x - y$ the graph $K - u - v$ is a cycle, so the third vertex is joined to x and y, i.e., $K = K_5$. If $n \geq 4$, let a, b, c, d be four consecutive vertices of the cycle $K - x - y$. $K - b - c$ is a cycle, hence one of a, d is joined to x but not to y, and the other to y but not to x. $n \geq 5$ ⇒ contradiction, $n = 4$ ⇒ $K_{3,3}$.

Minors

K_7 is a toroidal graph (it is embeddable on the torus), K_8 is not. What else is not? For the torus there is NO equivalent version of Kuratowski's characterization with a finite number of forbidden subdivisions. Any such characterization must lead to an infinite list.

A weaker concept: Minors.

Graph G is called a minor of graph H if G can be obtained from H by a series of deletions and edge contractions. Graph H is also called a G-minor

Example: K_5 is a minor of the Petersen graph P, but P does not contain a K_5-subdivision.

The Graph Minor Theorem

Theorem. (Robertson and Seymour, 1985-2005) In any infinite list of graphs, some graph is a minor of another.

Proof: more than 500 pages in 20 papers.

Corollary For any graph property that is closed under taking minors, there exists finitely many minimal forbidden minors.

Remark: Wagner's Theorem, stating that every non-planar graph contains either a K_5 or $K_{3,3}$-minor, can be (quite straightforwardly) deduced from Kuratowski's Theorem.

For embeddability on the projective plane, it is known that there are 35 minimal forbidden minors. For embeddability on the torus, we don't know the exact number of minimal forbidden minors; there are more than 800 known.