

Graphs & Algorithms II

Exercise Set 9

HS09

URL: <http://www.ti.inf.ethz.ch/ew/courses/GA09/>

Exercise 27

- Show that for any set P of n points in \mathbb{R}^2 there are at most $O(n^{3/2})$ pairs of points in $\binom{P}{2}$ that have Euclidean distance exactly one. (*Hint*: Show that the unit distance graph does not contain $K_{2,3}$ as a subgraph.)
- Show that for any set P of n points in \mathbb{R}^3 there are at most $O(n^{5/3})$ pairs of points in $\binom{P}{2}$ that have Euclidean distance exactly one.

Exercise 28

- Let $G = (V, E)$ be a graph with average vertex degree at least $2k$. Show that there exists a subgraph H in G in which every vertex has degree at least k .
- Let T be a tree on k vertices. Prove that $\frac{n(k-2)}{2} \leq \text{ex}(n, T) < nk$, for any $n \in \mathbb{N}$ such that $n \bmod (k-1) \equiv 0$.

Remark: The lower bound is conjectured to be tight.

Exercise 29

Consider the finite field \mathcal{F}_p for some prime number p and define a graph $G_p = (V_p, E_p)$ on the vertex set $V_p = \mathcal{F}_p^2 \setminus \{(0, 0)\}$. Two points (x, y) and (x', y') from V_p are connected by an edge in G_p if and only if they are distinct and $xx' + yy' \equiv 1 \pmod{p}$.

- Prove that G_p does not contain $K_{2,2}$ as a subgraph.
- Show that $|E_p| \geq \frac{1}{2}(p-1)(p^2-1)$.
- Conclude that $\text{ex}(n, K_{2,2}) = \Omega(n^{3/2})$.

Remark: Together with Homework 9 it follows that $\text{ex}(n, K_{2,2}) = \Theta(n^{3/2})$.

Homework 9

- Let $G = (V, E)$ be a graph which does not contain $K_{s,t}$ as a subgraph, for any $s, t \in \mathbb{N}$. Prove that $\sum_{v \in V} \binom{d(v)}{s} \leq (t-1) \binom{|V|}{s}$.
- Prove that for any $s, t \in \mathbb{N}$ there is some constant $c = c(s, t) \in \mathbb{R}$ (i.e., c depends on s and t but not on n) such that $\text{ex}(n, K_{s,t}) \leq cn^{2-1/s}$, for any $n \in \mathbb{N}$.

Homework due: 19.11.2009, 11:00AM.