Exercise 1 (Exercise 3 from Set 8)
Recall that we call a weak realization standard if the corresponding drawing of $G$ is standard, by which we mean that the edges are drawn as polygonal curves, every two intersect at finitely many points, and no three edges have a common intersection (where sharing a vertex does not count).

(a) Prove that if $(G, R)$ has a weak realization, then it also has a standard weak realization.

(b) Prove that if $(G, R)$ has a weak realization $W$ with finitely many edge intersections in which no three edges have a common intersection, then it also has a standard weak realization $W'$ with at most as many edge intersections as in $W$.

Exercise 2
Recall the following definitions from the lecture notes. For a string graph $G$, let $f_s(G)$ denote the minimum number of intersection points in a standard string representation of $G$, and let

$$f_s(n) := \max\{f_s(G) : \text{$G$ a string graph on $n$ vertices}\}.$$  

Similarly, for an abstract topological graph $(G, R)$ admitting a weak realization, let $f_w(G, R)$ be the minimum number of edge intersections in a standard weak realization of $(G, R)$, and

$$f_w(m) := \max\{f_w(G, R) : (G, R) \text{ weakly realizable, } |E(G)| = m\}.$$  

Prove that

$$f_s(n) = O(f_w(n^2) + n^2).$$

Exercise 3
Recall the proof from the lecture that shows the number of crossings in a standard weak realization can be reduced if an edge $e$ is crossed at least $2^m$ times. Let $\hat{e}$ be a contiguous segment of $e$ that is crossed at least once and is crossed an even number of times by each segment. Let $2n_f$ be the number of times an edge $f \neq e$ crossed $\hat{e}$. Recall that the procedure from the lecture reduces the number of crossings with $\hat{e}$ from $2 \sum_f n_f$ to at most $\sum_f n_f$. Give an example where the number of crossings with $\hat{e}$ is reduced to exactly $\sum_f n_f$.

Exercise 4
Let $G$ be a graph. Recall that a subset $S \subseteq V(G)$ is called a separator if there is a partition of $V(G) \setminus S$ into disjoint subsets $A$ and $B$ such that there are no edges between $A$ and $B$ and $|A|, |B| \leq \frac{1}{2}|V(G)|$.

Check that the above definition of a separator is equivalent to requiring all connected components of $G \setminus S$ to have at most $\frac{1}{2}|V(G)|$ vertices.
Exercise 5

(a) Show that every tree has a one-vertex separator.

(b) We could also define a $\beta$-separator, for $\beta \in (0, 1)$, by replacing $\frac{2}{3}$ in the above definition by $\beta$. Check that for $\beta < \frac{2}{3}$, there are trees with no one-vertex $\beta$-separator. From this point of view, the value $\frac{2}{3}$ is natural, but for applications, $\beta$-separators for any constant $\beta < 1$ would do as well.

Exercise 6

Show that the $m \times m$ square grid has no separator of size $cm$ for some constant $0 < c < 1$. Thus, the order of magnitude in the Lipton–Tarjan theorem cannot be improved.