Exercise 1

(a) Give the flip tree associated with the 3-vint below and indicate the rigid edges.

(b) Give a 3-vint which is associated with the flip tree below.

Exercise 2

In the lecture, we defined the crossing number $c(G)$ of a graph $G$ as the minimum number of crossings that occur in any embedding of $G$ into $\mathbb{R}^2$. In this exercise, we consider the rectilinear crossing number $\overline{c}(G)$ of a graph $G$: the minimum number of crossing that occur in any straight-line drawing of $G$. Note that choosing the positions of the vertices fixes the drawing and that $\overline{c}(G)$ is maximum if $G$ is a complete graph.

Hence, we define $\overline{c}(P)$ for any $P \subset \mathbb{R}^2$ with $n = |P|$ to be the number of crossings in the drawing obtained by embedding $K_n$ on $P$. Now let $\phi(n) = \min_{P \subset \mathbb{R}^2 \text{ in general position}} \overline{c}(P)$ be the minimum number of crossings in any straight-line drawing of $K_n$.

(a) Let $P \subset \mathbb{R}^2$ be a set of points in general position. What is the relation between $\overline{c}(P)$ and the number of convex quadrilaterals with vertices in $P$?

(b) We define $\psi(n) = \phi(n)/\binom{n}{4}$. Prove that there exists a constant $c$ such that $\lim_{n \to \infty} \psi(n) = c$. This constant is called the rectilinear crossing number and is known to be between 0.379972 and 0.380488.

*Hint: double-count the set $S$ of pairs $(p, q)$ where $p \in P$ and $q \in \binom{P - p}{4}$ is a convex quadrilateral on $P - p$.**
Exercise 3

Recall the Crossing Lemma from the lecture:

**Lemma 1.** Let $G$ be any graph on $v$ vertices and $e$ edges. If $e \geq 4v$, then

$$cr(G) \geq \frac{1}{64} \cdot \frac{e^3}{v^2}.$$ 

Show that the lemma is asymptotically tight in the following (broad) sense: for any $v$ and $e$, where $4v \leq e \leq \binom{v}{2}$, there exist graphs with $v$ vertices, $e$ edges and crossing number $O(e^3/v^2)$.

*Hint: take a $\sqrt{n} \times \sqrt{n}$ integer grid as a vertex set, slightly perturbed to be in general position.*