Exercise 1: Clique in Unit Disk Intersection Graphs

Consider a point set $P$ with corresponding intersection graph $G = (P, E)$ where $uv \in E$ if and only if $d(u, v) \leq 2$. We want to construct an algorithm to find a maximum clique in $G$ in polynomial time.

(a) Consider a maximum clique $C$ in $G$. Let $a, b \in C$ be two points which maximize distance $d(a, b)$. Prove that all points of $C$ are in the lens of $ab$.

(b) Prove that all points in a half-lens of $ab$ form a clique in $G$.

(c) Prove König’s Theorem:

Theorem 1 (König 1931). The cardinality of a maximum matching in a bipartite graph $G'$ is equal to the cardinality of a minimum vertex cover of $G'$.

(d) Give an algorithm to compute the size of a maximum clique of $G$ in polynomial time.

Exercise 2: Transversal Numbers

Let $\gamma$ be the smallest integer such that the following always holds: for every clique $C$ in a disk intersection graph, there exists a set of at most $\gamma$ points that, together, touch every disk in $C$. It is known that $\gamma = 4$.

(a) Suppose that we consider unit disk intersection graphs instead of disk intersection graphs and let $\gamma'$ be the constant corresponding to this restricted setting. Give a constant $c$ and prove that $\gamma' < c$.

(b) Prove that $\gamma \geq 3$.

Exercise 3: Clique in Disk Intersection Graphs

Consider the following algorithm for finding a clique in a disk intersection graph (where the representation is given):

1. for each pair $c, c'$ of cells of the arrangement:
   a) consider the set $D_{c,c'}$ of disks that contain $c$, or $c'$, or both – this set induces the complement of a bipartite graph
   b) find a maximum clique $C_{c,c'}$ in this graph
2. output the largest clique $C_{c,c'}$ among all pairs $c, c'$

From the fact that $\gamma = 4$, prove that the clique returned by the algorithm has size at least half that of the maximum clique.
Exercise 4: Gaps in the Proof

In the lecture on the kissing disks theorem, several parts of the proof were omitted. In this exercise, we prove these missing parts.

(a) Let $u, v$ and $w$ be the centers of three mutually tangent disks on the plane with radii $r_u, r_v$ and $r_w$, respectively. Prove formally that if we increase $r_u$ and decrease $r_v$ and $r_w$ so that the disks remain tangent, then $\angle uvw$ will decrease.

(b) Let $G$ be a maximal planar graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ for $n > 3$. For any $U \subseteq V(G)$, let $F(U)$ denote the set of all faces (including the outer face) of $G$ having at least one vertex that belongs to $U$. Prove that $|F(U)| > 2|U|$ for any $U$ with $1 \leq |U| \leq n-3$. 