Line graphs and edge coloring

A k-edge-coloring of a graph G is a labeling $f : E(G) \to S$, where $|S| = k$. The labels are called colors; the vertices of one color form a color class. A k-edge-coloring is proper if adjacent vertices have different labels. A graph is k-edge-colorable if it has a proper k-edge-coloring.

The edge-chromatic number (or chromatic index) of a loopless graph G is

$$\chi'(G) := \min\{k : G \text{ is } k\text{-colorable}\}.$$

A graph G is k-edge-chromatic if $\chi'(G) = k$.

Remarks. $\chi'(G) = \chi(L(G))$, so

$$\begin{array}{ccccc} \Delta(G) & \leq & \omega(L(G)) & & \\ & \leq & \chi'(G) & \leq & \Delta(L(G)) + 1 \\ & & & \leq & 2\Delta(G) - 1 \end{array}$$

Vizing's Theorem

Example. K_{2n}

Theorem. (König, 1916) For a bipartite graph G, $\chi'(G) = \Delta(G)$

Proposition. $\chi'(Petersen) = 4$.

Theorem. (Vizing, 1964) For a simple graph G,

$$\chi'(G) \leq \Delta(G) + 1.$$

Example. Fat triangle; $\chi'(G) = \Delta(G) + \mu(G)$.