## Connectivity

A graph $G$ is connected, if there is a $u, v$-path for every pair $u, v \in V(G)$ of vertices. Otherwise $G$ is disconnected.

Vertex $u$ is connected to vertex $v$ in $G$ if there is a $u, v$-path. The connection relation on $V(G)$ consists of the ordered pairs $(u, v)$ such that $u$ is connected to $v$.

**Claim.** The connection relation is an equivalence relation.

**Lemma.** Every $u, v$-walk contains a $u, v$-path.

The connected components of $G$ are its maximal connected subgraphs (i.e. the equivalence classes of the connection relation).

An isolated vertex is a vertex of degree 0. It is a connected component on its own, called trivial connected component.



**Proposition.** Every graph with $n$ vertices and $k$ edges has at least $n - k$ components.

A cut-edge or cut-vertex of a graph $G$ is an edge or a vertex whose deletion increases the number of components.

If $M \subseteq E(G)$, then $G - M$ denotes the graphs obtained from $G$ by the deletion of the elements of $M$; $V(G - M) = V(G)$ and $E(G - M) = E(G) \setminus M$. Similarly, for $S \subseteq V(G)$, $G - S$ obtained from $G$ by the deletion of $S$ and all edges incident with a vertex from $S$.
For $e \in E(G)$, $G - \{e\}$ is abbreviated by $G - e$.
For $v \in E(G)$, $G - \{v\}$ is abbreviated by $G - v$.

**Theorem.** An edge $e$ is a cut-edge iff it does not belong to a cycle.

## Strong Induction

**Theorem 1.** (Principle of Induction) Let $P(n)$ be a statement with integer parameter $n$. If the following two conditions hold then $P(n)$ is true for each positive integer $n$.

1. $P(1)$ is true.
2. For all $n > 1$, "$P(n-1)$ is true" implies "$P(n)$ is true".

**Theorem 2.** (Strong Principle of Induction) Let $P(n)$ be a statement with integer parameter $n$. If the following two conditions hold then $P(n)$ is true for each positive integer $n$.

1. $P(1)$ is true.
2. For all $n > 1$, "$P(k)$ is true for $1 \leq k < n$" implies "$P(n)$ is true".



## Bipartite graphs

A bipartition of $G$ is a specification of two disjoint independent sets in $G$ whose union is $V(G)$.

**Theorem.** (König, 1936) A graph $G$ is bipartite iff $G$ does not contain an odd cycle.

**Lemma.** Every closed odd walk contains an odd cycle.

The union $G_1 \cup \cdots \cup G_k$ of graphs $G_1, \ldots, G_k$ is the graph with vertex set $V(G_1) \cup \cdots \cup V(G_k)$ and edge set $E(G_1) \cup \cdots \cup E(G_k)$.

**Theorem.** The complete graph $K_n$ is the union of $k$ bipartite graphs iff $n \leq 2^k$.

## Eulerian circuits and trails

A graph is Eulerian if it has a closed trail containing all its edges. A graph is called even if all of its vertices have even degree.

**Theorem.** A connected graph $G$ is Eulerian iff it is even.

> **Lemma.** If every vertex of a graph $G$ has degree at least 2, then $G$ contains a cycle.

**Corollary of the proof.** Every even graph decomposes into cycles.

**Theorem.** A connected nontrivial graph with exactly $2k$ vertices of odd degree decomposes into $\max\{k, 1\}$ trails.

*Remark.* The theorem is "best possible", i.e. a decomposition into *less* than $\max\{k, 1\}$ trails is not possible.



## Further proofs via extremality

**Proposition.** Let $k \geq 2$. If $\delta(G) \geq k$ for a simple graph $G$, then $G$ contains a cycle of length at least $k + 1$.

**Proposition.** Every graph with a non-loop edge has at least two vertices that are not cut-vertices.



## Proof techniques

- (Strong) induction
- Extremality
- Double counting



## Neighborhoods and degrees...

The neighborhood of $v$ in $G$ is $N_G(v) = \{w \in V(G) : vw \in E(G)\}$.
The degree of a vertex $v$ in graph $G$ is $d_G(v) = |N_G(v)|$.

The maximum degree of $G$ is $\Delta(G) = \max_{v \in V(G)} d(v)$

The minimum degree of $G$ is $\delta(G) = \min_{v \in V(G)} d(v)$

$G$ is regular if $\Delta(G) = \delta(G)$
$G$ is $k$-regular if the degree of each vertex is $k$.

The order of graph $G$ is $n(G) = |V(G)|$.
The size of graph $G$ is $e(G) = |E(G)|$.

**Handshaking Lemma.** For any graph $G$,

$$\sum_{v \in V(G)} d(v) = 2e(G).$$

**Corollary.** Every graph has an even number of vertices of odd degree. No graph of odd order is regular with odd degree.

**Corollary.** In a graph $G$ the average degree is $\frac{2e(G)}{n(G)}$ and hence $\delta(G) \leq \frac{2e(G)}{n(G)} \leq \Delta(G)$.

**Corollary.** A $k$-regular graph with $n$ vertices has $kn/2$ edges.