Double counting and bijections I

Handshaking Lemma. For any graph G,

$$\sum_{v \in V(G)} d(v) = 2e(G).$$

Corollary. Every graph has an even number of vertices of odd degree. No graph of odd order is regular with odd degree.

Corollary. In a graph G the average degree is $\frac{2e(G)}{n(G)}$ and hence $\delta(G) \le \frac{2e(G)}{n(G)} \le \Delta(G)$.

Corollary. A k-regular graph with n vertices has $kn/2$ edges.

The k-dimensional hypercube Q_k

$V(Q_k) = \{0,1\}^k$

$E(Q_k) = \{xy : x$ and y differ in exactly one coordinate$\}$

Properties.

- $n(Q_k) = 2^k$
- Q_k is k-regular
- $e(Q_k) = k2^{k-1}$
- Q_k is bipartite
- The number of j-dimensional subcubes (subgraphs isomorphic to Q_j) of Q_k is $\binom{k}{j}2^{k-j}$.

Double counting and bijections II

Proposition. If $k > 0$, then a k-regular bipartite graph G has the same number of vertices in each bipartite set.

Proof. Double count the edges of G.

Claim. The Petersen graph contains ten 6-cycles.

Proof. Bijection between 6-cycles and claws. (A claw is a $K_{1,3}$.)

Extremal problems* — Examples

Vague description: An extremal problem asks for the maximum or minimum value of a parameter over a class of objects (graphs, in most cases).

Proposition. The minimum number of edges in a connected graph is $n-1$.

Proposition. G is simple n-vertex graph with $\delta(G) \ge \lfloor n/2 \rfloor$, then G is connected.

Remark. The above proposition is *best possible*, as shown by $K_{\lfloor n/2 \rfloor} + K_{\lceil n/2 \rceil}$.

Graph $G + H$ is the disjoint union (or sum) of graphs G and H. For an integer m, mG is the graph consisting of m disjoint copies of G.

*My favorite topic

Extremal Problems

graph property	graph parameter	type of extremum	value of extremum
connected	$e(G)$	minimum	$n-1$
simple, disconnected	$\delta(G)$	maximum	$\left\lfloor \frac{n}{2} \right\rfloor - 1$
simple, K_3-free	$e(G)$	maximum	$\left\lfloor \frac{n^2}{4} \right\rfloor$

Triangle-free subgraphs

Theorem. (Mantel, 1907) The maximum number of edges in an n-vertex triangle-free simple graph is $\lfloor \frac{n^2}{4} \rfloor$.

Proof.

(i) *There is* a triangle-free graph with $\lfloor \frac{n^2}{4} \rfloor$ edges.

(ii) If G is a triangle-free graph, then $e(G) \leq \lfloor \frac{n^2}{4} \rfloor$.

Proof of (ii) is with extremality. (Look at the neighborhood of a vertex of maximum degree.)

Example of a wrong proof of (ii) by induction.

Bipartite subgraphs

Theorem. Every loopless graph G has a bipartite subgraph with at least $e(G)/2$ edges.

Proof # 1. Algorithmic. (Start from an arbitrary bipartition and move over a vertex whose degree in its own part is *more* than its degree in the other part. Iterate.)

Proof # 2. Extremality. (Consider a bipartite subgraph H with the *maximum number of edges*, prove that $d_H(v) \geq d_G(v)/2$ for every vertex $v \in V(G)$ and use the Handshaking Lemma.)

Remark 1. *Maximum vs. maximal.* Algorithmic proof *not* necessarily ends up in bipartite subgraph with maximum number of edges.

Remark 2. The constant multiplier $\frac{1}{2}$ of $e(G)$ in the Theorem is best possible. *Example:* K_n.

Directed graphs

A directed graph (or digraph) is a triple consisting of a vertex set $V(G)$, edge set $E(G)$, and a function assigning each edge an ordered pair of vertices.

For an edge $e = (x, y)$, x is the tail of e, y is its head.

By path and cycle in a directed graph we always mean directed path and directed cycle.

A directed graph is weakly connected if the underlying undirected graph is connected; it is strongly connected or strong if there is a u, v-path for any vertex u and v.

The out-neighborhood of v in G is $N_G^+(v) = \{w \in V(G) : (v, w) \in E(G)\}$.
The out-degree of v is $d_G^+(v) = |N_G^+(v)|$.

The in-neighborhood of v in G is $N_G^-(v) = \{w \in V(G) : (w, v) \in E(G)\}$.
The in-degree of v is $d_G^-(v) = |N_G^-(v)|$.

Déjà vu

Directed Handshaking. In a digraph G,

$$\sum_{v \in V(G)} d^+(v) = e(G) = \sum_{v \in V(G)} d^-(v).$$

A digraph is Eulerian if it has a directed Eulerian circuit, i.e. a closed directed trail containing all edges.

Theorem. A digraph is Eulerian iff $d^+(v) = d^-(v)$ for each vertex v and the underlying undirected graph has at most one nontrivial component.

Proof. Similar to the undirected case. Think it over.

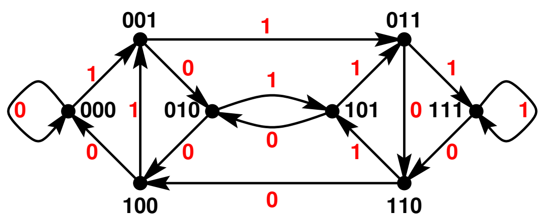
Application. de Bruijn cycles

de Bruijn cycles via de Bruijn digraphs

$V(D_n) = \{0, 1\}^{n-1}$

$E(D_n) = \{(a, b) : a_{i+1} = b_i$ for $i = 1, \ldots n - 2\}$

Label each edge (a, b) with b_{n-1}.



Theorem. The digraph D_n is Eulerian. Moreover, the edge labels on the edges in any Eulerian circuit of D_n form a cyclic arrangement in which the 2^n consecutive segments of length n are distinct.

Kings

An orientation of an undirected graph G is a digraph D obtained by choosing an orientation for each edge of G. An oriented graph is an orientation of a simple graph. A tournament is an orientation of the complete graph.

A vertex $v \in V(G)$ of a digraph G is a king if every vertex of G is reachable from v on a path of length at most 2.

Proposition. Every tournament has a king.

Proof. Extremality. (Look at a vertex of maximum out-degree.)